The Haldane bosonisation scheme and metallic states of interacting fermions in d spatial dimensions

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We consider the Haldane bosonisation scheme in d spatial dimensions as applied to a realistic model of interacting fermions in $d = 2$ and unequivocally demonstrate failure of this scheme in $d > 1$, specifically in $d = 2$. In addition to tracing back this failure to its origin, we show that *nothing* as regards the true metallic state of the model under consideration is known with any degree of certainty.

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I. INTRODUCTION

Properties of high-temperature (hereafter hight-T_c) cuprate superconductors in their normal states indicate these to be non-Fermi liquids (NFLs) (Anderson 1997), yet after one decade of intensive efforts by researchers, correspondence of a NFL metallic state to a microscopic model (without magnetic impurities) in spatial dimensions d greater than one has proved elusive: barring non-uniform and superconducting states (at absolute zero of temperature), these efforts have consistently borne out the existence of solely Fermi-liquid (FL) metallic states in $d > 1$ (for a review see Farid 1999a). In consequence, a specific category of NFLs, namely that of Luttinger's, has prevailed to be considered viable only in $d = 1$. The existence of NFLs, specifically Luttinger liquids (LLs) (Haldane 1980, 1981) in $d > 1$, as postulated by Anderson (1988, 1989, 1990a,b, 1992, 1993; Anderson and Ren 1990), not only would, according to Anderson and co-workers, provide the basis for a unified explanation of the anomalous normal-state properties of the high-T_c cuprates, but also would account for the superconductivity-pairing mechanism in these materials (Wheatley, Hsu, Anderson 1988a,b, Anderson 1991a,b, 1992, Chakravarty, Sudbø, Anderson and Strong 1993, Anderson 1997). On the basis of their theoretical studies — which rely upon Haldane's (1992) non-perturbative bosonisation scheme for gap-less degrees of freedom of interacting fermions in arbitrary spatial dimensions d — of a microscopic model of interacting electrons in $d = 2$, originally put forward by Houghton and Marston (1993), Houghton, Kwon and Marston (HKM) (1994) and Houghton, Kwon, Marston and Shankar (HKMS) (1994) have arrived at the conclusion that (see also Kwon, Houghton and Marston 1995) i) for fermions interacting through short-range and the long-range Coulomb interaction function, the metallic state in $d = 2$ is a FL and that, in conformity with Bares and Wen (1993), ii) for super long-range interacting fermions the metallic state is a NFL. [1] In this work we unequivocally demonstrate that conclusions (i) and (ii) are biased in consequence of insufficiently detailed considerations and application of an unjustified approximation employed by HKM (1994) and HKMS (1994). More importantly, we demonstrate that in its present form the Haldane (1992) bosonisation method suffers from a fundamental shortcoming in $d > 1$, rendering any conclusion obtained through its application indefensible. We thus conclude that on the basis of the available

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[1] To be precise, Bares and Wen (1993) refer to this state as a "$Z_{k_F} = 0$ Fermi liquid", where Z_{k_F} stands for the value of the jump in the momentum distribution function $\mathsf{n}(k)$ at Fermi momentum k_F.

knowledge, the true metallic state of the model under consideration *cannot* be considered to be known with any degree of certainty.

The organisation of this work is as follows. In Section II we deduce some leading terms in the asymptotic series of $\delta G(\omega)$, the difference between the interacting and non-interacting Green functions, for small values of energy ω. Here we explicitly deal with the conventional FLs in two and three spatial dimensions. In Section III we briefly describe some aspects of the Haldane bosonisation scheme that are relevant to our considerations in this work and obtain, directly from the energy-momentum $(\omega, \boldsymbol{k})$ representation of G within this scheme, the coefficient of the $1/\omega$ term in the asymptotic series for small ω of $\delta G(\omega)$. The sign of this coefficient turns out to be negative which is opposite to that corresponding to conventional FLs. Here we establish that this property is common to a class of un-conventional FLs where the Fermi energy of the interacting system coincides with that of the non-interacting system. Our subsequent analyses bring out that in fact according to the Haldane bosonisation scheme, *all* metallic systems in $d > 1$ must belong to this category of unconventional FLs (that is, this holds true independent of the nature and strength of the interaction amongst the fermions). We devote Section IV to a discussion of the long-distance behaviour of the advanced part of the single-particle Green function, that is $G(\boldsymbol{x}, t = 0^-)$. Here we expose a close relationship that exists between this behaviour and some smoothness property of the momentum-distribution function $\mathsf{n}(\boldsymbol{k})$ and specifically determine the condition to be met by $\mathsf{n}(\boldsymbol{k})$ in order for the anomalous exponent of $G(\boldsymbol{x}, t = 0^-)$ to be vanishing, that is one that renders the large-$\|\boldsymbol{x}\|$ behaviour of $G(\boldsymbol{x}, t = 0^-)$ to be FL-like. Our finding in this section with regard to the just-indicated FL-like behaviour of $G(\boldsymbol{x}, t = 0^-)$ at large $\|\boldsymbol{x}\|$ within the Haldane bosonisation scheme for $d > 1$, leads us to the conclusion that this condition must be built into the structure of this scheme. Consequently, in Section V we consider the algebraic structure underlying the Haldane programme and establish that the Kac-Moody algebra concerning some current operators, which is basic to this structure, neglects in $d > 1$ some contributions that are the more significant the greater the deviation from unity is of the amount of discontinuity in $\mathsf{n}(\boldsymbol{k})$ at the Fermi momentum. In other words, in $d > 1$ the just-indicated Kac-Moody algebra turns out to give a correct account of the physical processes in the system *only* when the fermions are strictly non-interacting. This finding clarifies the reason underlying our two observations mentioned above, namely that within the framework of the Haldane bosonisation scheme the Fermi energy of the interacting system coincides with that of the *non-interacting* system and that the anomalous dimension corresponding to the large-$\|\boldsymbol{x}\|$ behaviour of $G(\boldsymbol{x}, t = 0^-)$ is vanishing, a property that interacting and non-interacting FLs share. In Section VI we present a brief description of the existing alternative realisations of the original Haldane bosonisation approach in $d > 1$ and argue that none of them is capable of describing NFL metallic states in $d > 1$. We conclude our work by a summary and some remarks.

II. PRELIMINARIES

Classification of metallic states is easiest accomplished through that of the self-energy Σ in the energy-momentum representation, for ω and $\boldsymbol{k}$ close to the Fermi energy and Fermi surface (FS), respectively. For simplicity, consider a uniform system of spin-less fermions in d spatial dimensions interacting through the isotropic two-body potential v. For k and ω in the vicinities of the Fermi wave number k_F and Fermi energy ω_F (both of which we identify with zero when no confusion can arise), respectively, FLs (un-conventional ones included) comprise those metallic states whose corresponding $\Sigma(k, 0)$ and $\Sigma(0, \omega)$ are continuously-differentiable functions of k and ω in neighbourhoods of $k = 0$ and $\omega = 0$, respectively (for details see Farid 1999a). In $d = 3$, self-energies of *conventional* FLs for $\omega \to 0$ and small k asymptotically behave like [2] $\Sigma(k, \omega) \sim \Sigma(k, 0) + \beta_k\, \omega - i\, \alpha_k\, \mathrm{sgn}(\omega)\, \omega^2$ (Hugenholtz 1957, DuBois 1959, Luttinger 1961) with $\alpha_k \geq 0$ (for β_k see below). The same expression holds for *conventional* FLs in $d = 2$ when $k \neq 0$; when $k = 0$, however, ω^2 in this expression is to be replaced by $-\omega^2 \ln|\omega|$ (Hodges, Smith and Wilkins 1971, Bloom 1975, Fujimoto 1990, Fukuyama, Narikio and Hasegawa 1991). It is for our further considerations relevant to mention that *all* FLs have in common that $\mathrm{Im}\Sigma(k, \omega) = o(\omega)$, where $f(\omega) = o\big(g(\omega)\big)$ signifies that $f(\omega)/g(\omega) \to 0$ for $\omega \to 0$; thus $|\omega|^{1+\alpha}$ with $\alpha > 0$ qualifies to be denoted by $o(\omega)$. In consequence of this property, making use of the Kramers-Krönig relation, i.e. the one expressing $\mathrm{Re}\Sigma(k, \omega)$ in terms of a principal-value integral of $\mathrm{Im}\Sigma(k, \omega)$, while invoking the requirement of stability of the *interacting* system, it can be rigorously proved (Farid 1999b) that in the

[2] We point out that here and in what follows we restrict our considerations to the leading ω-dependent asymptotic contributions to $\mathrm{Re}\Sigma(k, \omega)$ and $\mathrm{Im}\Sigma(k, \omega)$, for $\omega \to 0$, taken as independent functions; otherwise, taking into account, for instance, $-i\alpha_k\, \mathrm{sgn}(\omega)\, \omega^2$, would require to take into account the *real* contribution $\gamma_k\, \omega^2 \ln|\omega|$ which is asymptotically more dominant than $i\, \alpha_k\, \mathrm{sgn}(\omega)\, \omega^2$.

asymptotic series expansion of $\Sigma(k,\omega)$, for $\omega \to 0$, pertaining to an *interacting* FL system, a term must occur of the form $\beta_k\,\omega$ (*c.f.* the above-presented expression) where $\beta_k < 0$ (strictly negative). It follows that an *interacting* system for which $\beta_k = 0$ for *any* value of k (with k close to zero so that a separation of $\Sigma(k,\omega)$ into separate functions of k and ω can be rigorously effected), *cannot* be a FL.

The self-energy Σ is related to the single-particle Green function G through the Dyson equation (Fetter and Walecka 1971), which in the ω, k-representation reads $G(k,\omega) = G_0(k,\omega) + \delta G(k,\omega)$, where $\delta G(k,\omega) := G_0(k,\omega)\Sigma(k,\omega)G(k,\omega)$, with $G_0(k,\omega)$ and $G(k,\omega)$ the single-particle Green functions pertaining to the non-interacting and interacting systems, respectively. From this equation the following expression is readily obtained

$$\delta G(k,\omega) = \frac{\Sigma(k,\omega)}{[G_0^{-1}(k,\omega) - \Sigma(k,\omega)]G_0^{-1}(k,\omega)}. \tag{1}$$

In what follows for simplicity, but without loss of generality, we often consider the case corresponding to $k = 0$. Consequently, when $k = 0$, we suppress k in the arguments of the pertinent functions; thus for instance $\Sigma(\omega)$ will denote $\Sigma(0,\omega)$. We note that for *conventional* FLs holds $\Sigma(0) \neq 0$. [3] Exception concerns the case of non-interacting fermions for which $\Sigma(k,\omega) \equiv 0$.

From the above-presented asymptotic expression for the self-energy of FLs, making use of Eq. (1) and $G_0^{-1}(\omega) = \omega + \Sigma(0)$, one readily obtains [4]

$$\delta G(\omega) \sim \frac{1/(1-\beta_0)}{\omega} - i\,\frac{\alpha_0}{(1-\beta_0)^2}\,\mathrm{sgn}(\omega)\,\phi(\omega) - \frac{1}{\Sigma(0)}, \quad \omega \to 0, \tag{2}$$

where $\phi(\omega) \equiv -\ln|\omega|$ in $d = 2$ and $\phi(\omega) \equiv 1$ in $d = 3$. In Eq. (2), $\beta_0 = 1 - 1/Z_0$ where $Z_0 \in (0,1)$ is the amount of jump discontinuity in the momentum-distribution function $\mathsf{n}(k)$ of the system at $k = 0$ (at places in the following we denote Z_0 by $Z_{\boldsymbol{k}_F}$).

III. THE HALDANE BOSONISATION TECHNIQUE AND ITS APPLICATION TO A TWO-DIMENSIONAL MODEL OF INTERACTING FERMIONS

Starting from a many-body Hamiltonian for fermions interacting through a two-body isotropic potential, integrating out the high-energy degrees of freedom of the system by means of the renormalisation-group technique in the momentum space, Houghton and Marston (1993) have arrived at an effective Hamiltonian which up to regular terms — i.e. those which are *irrelevant* from the point of view of the renormalisation-group approach — can be expressed in terms of current operators (see Eq. (32) below and text following it). Employing Haldane's (1992) scheme in dealing with low-energy degrees of freedom of interacting fermions (i.e. making use of Haldane's 'patching' scheme of a narrow band encompassing the FS), Houghton and Marston (1993), similar to Haldane (1992), establish (see Section V however) that up to some error terms — which would be made small by an appropriate choice of the 'patching' parameters — the mentioned current operators form a Kac-Moody algebra (see Brink and Hanneaux 1988, Goddard and Olive 1986). This renders the reduced problem, as described by the indicated effective Hamiltonian, exactly solvable through the process of bosonisation which amounts to introducing a set of bosonic operators in terms of which the current operators can be expressed. [5]

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[3] In the event that $\Sigma(0) = 0$, the Fermi energies of the interacting and non-interacting systems coincide, i.e. $\omega_F = \omega_F^0$. For fermions interacting through a hard-core potential of range a in $d = 3$, Galitskii (1958) obtains $\omega_F = \omega_F^0\big(1 + 4k_F a/[3\pi] + 4(11 - 2\ln 2)(k_F a)^2/[15\pi^2]\big)$. Further, according to the Seitz theorem (Seitz 1940, pp. 343 and 344; Mahan 1981, Ch. 5) one has $\omega_F = E_0(\rho_0) + \rho_0 \mathrm{d}E_0(\rho_0)/\mathrm{d}\rho_0$, where $E_0(\rho_0)$ stands for the ground-state total energy per particle and ρ_0 the homogeneous ground-state number density. Using the Gell-Mann and Brueckner (1957) expression for $E_0(\rho_0)$, valid for high densities of fermions interacting through the long-range Coulomb potential in $d = 3$, or expressions derived from quantum Monte-Carlo calculations for a wide range of densities, one can deduce that $\omega_F \neq \omega_F^0$.

[4] Neglect of $\Sigma(\omega)$ in the denominator of Eq. (1), as done by HKM (1994) and HKMS (1994), implies the following relation, to be compared with that in Eq. (2): $\delta G(\omega) \sim \Sigma(0)/\omega^2 + \beta_0/\omega - i\alpha_0\,\mathrm{sgn}(\omega)\,\phi(\omega)$, $\omega \to 0$.

[5] For completeness, we mention that the total bosonised Hamiltonian separates into two contributions, one involving the symmetric (or *charge*) and the other anti-symmetric (or *spin*) combination of the bosonic operators associated with two different spin states. In the treatment that we consider in the present work, the *spin* part of the Hamiltonian is neglected. This amounts to the assumption that excitations in the spin sector are of higher energy than those in the charge sector.

Although through application of the bosonisation technique one would in principle exactly solve the many-body problem in the low-energy region of the excitation spectrum (see Section V however), technical difficulties hinder exact determination of, e.g., $\Sigma(k,\omega)$, for small $|\omega|$, from the exact solution. This is owing to the fact that the single-particle Green function pertaining to the effective Hamiltonian, which we for clarity denote by $G_{\rm f}$ (subscript f making explicit that this Green function pertains to the actual *fermion* system) in the $\boldsymbol{k},\omega$ space is to be obtained from the Fourier transform with respect to the reciprocal variables $\boldsymbol{x},t$ (space-time) of an expression in which the $\boldsymbol{x},t$-representation of the exact Green function of the bosonic problem, which we denote by $G_{\rm b}$, is exponentiated. It should be noted that in the context of Haldane's (1992) bosonisation scheme, one obtains $G_{\rm b}(\boldsymbol{k},\omega;S)$ with S indicating the FS 'patch' to whose corresponding 'squat box' (Houghton and Marston 1993) $\boldsymbol{k}$ pertains. In d dimensions, S is a $(d-1)$-dimensional flat surface of linear size Λ, thus covering an area of extent Λ^{d-1}. The 'squat box' associated with S is defined as the region around $\boldsymbol{k}_F(S)$, the Fermi wave-vector at the centre of S, which consists of *all* points of 'patch' S translated by $\gamma\,\boldsymbol{n}_S$ with $\gamma\in[-\lambda/2,\lambda/2]$, $\lambda\ll\Lambda$; here $\boldsymbol{n}_S$ denotes the outward unit vector normal to S. For definiteness, let $\mathcal{F}_{\boldsymbol{k},\omega}[f]=:g(\boldsymbol{k},\omega)$ denote the Fourier transform of $f(\boldsymbol{x},t)$ to the $\boldsymbol{k},\omega$ domain and $\mathcal{F}^{-1}_{\boldsymbol{x},t}[g]\equiv f(\boldsymbol{x},t)$ its inverse. Let further [6] $\tilde{\mathcal{F}}^{-1}_{\boldsymbol{x},t}[g]:=\mathcal{F}^{-1}_{\boldsymbol{x},t}[g]-f(\mathbf{o},0)$. For $d=2$, $G_{\rm f}(\boldsymbol{k},\omega;S)$ is to be obtained from (HKM 1994, HKMS 1994)

$$G_{\rm f}(\boldsymbol{k},\omega;S)=\mathcal{F}_{\boldsymbol{k},\omega}\Big[\frac{\Lambda}{(2\pi)^2}\,\frac{\exp\big(i\boldsymbol{k}_F(S)\cdot\boldsymbol{x}\big)}{\boldsymbol{x}\cdot\boldsymbol{n}_S-t+i\,\eta\,\mathrm{sgn}(t)}\,\exp\big(\frac{2\pi i}{\Omega^2}\tilde{\mathcal{F}}^{-1}_{\boldsymbol{x},t}[\delta G_{\rm b}]\big)\Big],\quad |\boldsymbol{x}\cdot\boldsymbol{\tau}_S|\ll 1/\Lambda, \tag{3}$$

where $\Omega:=\Lambda(L/[2\pi])^2$ with L the (macroscopic) linear size of the square in which the system is confined and $\boldsymbol{\tau}_S$ is the unit vector normal to $\boldsymbol{n}_S$; $\delta G_{\rm b}$ denotes deviation of the full bosonic Green function from the free one. Note in passing that since each FS 'patch' is flat, $\boldsymbol{k}_F(S)=k_F(S)\,\boldsymbol{n}_S$. The expression in Eq. (3) makes explicit the complexity of calculation of $G_{\rm f}(\boldsymbol{k},\omega;S)$ from $G_{\rm b}(\boldsymbol{k},\omega;S)$.

For our following considerations it is useful to define

$$\Xi:=\exp\big(\frac{-2\pi i}{\Omega^2}\mathcal{F}^{-1}_{\boldsymbol{x}=\mathbf{o},t=0}[\delta G_{\rm b}]\big). \tag{4}$$

It is important to realise that $\Xi\neq 0$, for $\Xi=0$ implies $G_{\rm f}(\boldsymbol{k},\omega;S)\equiv 0$ which is meaningless. Following the above definition for Ξ, for $G_{0;{\rm f}}(\boldsymbol{k},\omega;S)$ and $\delta G_{\rm f}(\boldsymbol{k},\omega;S)$ in $G_{\rm f}(\boldsymbol{k},\omega;S)\equiv G_{0;{\rm f}}(\boldsymbol{k},\omega;S)+\delta G_{\rm f}(\boldsymbol{k},\omega;S)$ (compare with $G_0(k,\omega)$ and $\delta G(k,\omega)$ introduced above) we have [7]

$$G_{0;{\rm f}}(\boldsymbol{k},\omega;S):=\mathcal{F}_{\boldsymbol{k},\omega}\Big[\frac{\Lambda}{(2\pi)^2}\,\frac{\exp\big(i\boldsymbol{k}_F(S)\cdot\boldsymbol{x}\big)}{\boldsymbol{x}\cdot\boldsymbol{n}_S-t+i\,\eta\,\mathrm{sgn}(t)}\Big],\quad |\boldsymbol{x}\cdot\boldsymbol{\tau}|\ll 1/\Lambda, \tag{5}$$

$$\begin{aligned}\delta G_{\rm f}(\boldsymbol{k},\omega;S):&=(\Xi-1)\,G_{0;{\rm f}}(\boldsymbol{k},\omega;S)\\ &+\Xi\,\mathcal{F}_{\boldsymbol{k},\omega}\Big[\frac{\Lambda}{(2\pi)^2}\,\frac{\exp\big(i\boldsymbol{k}_F(S)\cdot\boldsymbol{x}\big)}{\boldsymbol{x}\cdot\boldsymbol{n}_S-t+i\,\eta\,\mathrm{sgn}(t)}\big\{\exp\big(\frac{2\pi i}{\Omega^2}\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\rm b}]\big)-1\big\}\Big],\quad |\boldsymbol{x}\cdot\boldsymbol{\tau}|\ll 1/\Lambda.\end{aligned} \tag{6}$$

It is readily verified that for $\boldsymbol{k}-\boldsymbol{k}_F(S)$ parallel to $\boldsymbol{n}_S$ (i.e. for $\boldsymbol{k}-\boldsymbol{k}_F(S)=k_\parallel\,\boldsymbol{n}_S$), with the range of the $\boldsymbol{x}$-integration along $\boldsymbol{\tau}_S$ restricted to interval $(-\pi/\Lambda,\pi/\Lambda)$, one has

$$G_{0;{\rm f}}(\boldsymbol{k},\omega;S)=\frac{1}{\omega-k_\parallel+i\,\eta\,\mathrm{sgn}(\omega)},\quad \eta\downarrow 0, \tag{7}$$

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[6] One can equivalently write $\tilde{\mathcal{F}}^{-1}_{x,t}[g]:=\mathcal{F}^{-1}_{x,t}[g]-\mathcal{F}^{-1}_{0,0}[g]$. We point out that in the case at hand, in particular for $d=2$, it can be shown that $\delta G_{\rm b}(x,t;S)$ is continuous at both $x=0$ and $t=0$ so that $\mathcal{F}^{-1}_{0,0}[\delta G_{\rm b}]$ is *not* ambiguous.

[7] We point out that our definition of $\delta G_{\rm f}(k,\omega;S)$ differs from that in the works by HKM (1994) and HKMS (1994). These authors employ the first-order expansion $\exp\big(2\pi i\Omega^{-2}\tilde{\mathcal{F}}^{-1}_{x,t}[\delta G_{\rm b}]\big)\approx 1+2\pi i\Omega^{-2}\tilde{\mathcal{F}}^{-1}_{x,t}[\delta G_{\rm b}]$ and tacitly dispose of the contribution $\mathcal{F}^{-1}_{0,0}[\delta G_{\rm b}]$ associated with the latter $\tilde{\mathcal{F}}^{-1}_{x,t}[\delta G_{\rm b}]$; thus in these author's work $\delta G_{\rm f}(k,\omega;S)\approx\mathcal{F}_{k,\omega}\big[(2\pi)^{-2}\Lambda\,\big\{\exp\big(ik_F(S)\cdot x\big)/[x\cdot n_S-t+i\,\eta\,\mathrm{sgn}(t)]\big\}\,2\pi i\Omega^{-2}\mathcal{F}^{-1}_{x,t}[\delta G_{\rm b}]\big]$, whereas using the same linear expansion we have $\delta G_{\rm f}(k,\omega;S)\approx\big(\Xi-1\big)\,G_{0;{\rm f}}(k,\omega;S)+\Xi\mathcal{F}_{k,\omega}\big[(2\pi)^{-2}\Lambda\,\big\{\exp\big(ik_F(S)\cdot x\big)/[x\cdot n_S-t+i\,\eta\,\mathrm{sgn}(t)]\big\}\,2\pi i\Omega^{-2}\mathcal{F}^{-1}_{x,t}[\delta G_{\rm b}]\big]$. Therefore, for properties which are associated with interaction, our (first-order) results can be directly compared with those by HKM (1994) and HKMS (1994) through dividing our results by Ξ, which, as we have indicated, is non-vanishing.

so that for $k_\parallel = 0$ the first term on the right-hand side (RHS) of Eq. (6) is seen to behave like $(\Xi-1)/\omega$ as $\omega \to 0$. Thus *if* for $\omega \to 0$ the leading asymptotic contribution due to the second term on the RHS of Eq. (6) is sub-dominant with regard to $1/\omega$, a comparison with Eq. (2) reveals that for conventional FLs must hold [8] $\Xi = 1/(1-\beta_0)+1 \equiv Z_{\boldsymbol{k}_F}+1$. For unconventional FLs, with $\Sigma(0) = 0$, on the other hand, $\Xi = \beta_0/(1-\beta_0)+1 \equiv Z_{\boldsymbol{k}_F}$ (see Eq. (11) below). It is important to realise that these relationships do *not* apply to NFLs, for our inference has been based upon Eq. (2) above and Eq. (11) below which are specific to FLs. In this connection we mention that for LLs the leading contribution to $\mathrm{Re}\delta G(\omega)$, which at the same time is the leading contribution to $\delta G(\omega)$, is *exactly* equal to $-1/\omega$, and for marginal FLs (Varma, *et al.*, 1989, 1990, Littlewood and Varma 1991), depending on whether $\Sigma(0) = 0$ or $\Sigma(0) \neq 0$, one has (exactly) $-1/\omega$ and $[-\Sigma(0)/\beta_0]/(\omega \ln|\omega|)$, $\omega \to 0$, respectively; here β_0 is the coefficient of $\omega \ln|\omega|$ in the asymptotic expansion of $\Sigma(\omega)$ for $\omega \to 0$. Since $\Xi \neq 0$, in cases where the leading asymptotic contribution to $\delta G(\omega)$ is equal to $-1/\omega$, for $\omega \to 0$, the leading asymptotic contribution of the second term on the RHS of Eq. (6) must be $-\Xi/\omega$, for $\omega \to 0$, thus cancelling the Ξ/ω contribution due to the first term. As we shall discuss below, within the approximation where $\exp\left(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\mathrm{b}}]\right) \approx 1+2\pi i\Omega^{-2}\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\mathrm{b}}]$, our explicit calculations establish that the second term on the RHS of Eq. (6) scales like $\ln^2|\omega|$ for $\omega \to 0$ so that, provided the latter approximation be justified, we are led to the conclusion that under *no* circumstance (i.e. independent of the nature of the particle-particle interaction) the system under consideration can be either a LL or a marginal FL. It is interesting to note that $\ln^2|\omega|$ does not coincide with any of the known next-to-leading asymptotic results (presented above) corresponding to $\delta G(\omega)$. This aspect may be viewed as indicating the inadequacy of the first-order approximation $\exp\left(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\mathrm{b}}]\right) \approx 1+2\pi i\Omega^{-2}\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\mathrm{b}}]$ as well as that of the Haldane (1992) bosonisation scheme in $d > 1$.

A. Simplified and approximate approaches

Due to the mentioned complexity of calculation of $G_{\mathrm{f}}(\boldsymbol{k},\omega;S)$, HKM (1994) and HKMS (1994) employ two expansions of which one in certain limits is in principle rigorous. [9] It is therefore the second of these expansions that will concern us most in our following considerations. As for the first expansion, in principle rigorous in certain limits, HKM (1994) and HKMS (1994) proceed from the exact expression for the proper self-energy of the bosonic problem, namely

$$\Sigma_{\mathrm{b}}(\boldsymbol{q},\omega;S) = \frac{2\Lambda^{d-1}\boldsymbol{n}_S\cdot\boldsymbol{q}}{(2\pi)^d}\,W(\boldsymbol{q},\omega), \qquad W(\boldsymbol{q},\omega):=\frac{v(\boldsymbol{q})}{1+v(\boldsymbol{q})\chi_0(\boldsymbol{q},\omega)}, \tag{8}$$

where $v(\boldsymbol{q})$ stands for the Fourier transform of the two-body interaction potential and $\chi_0(\boldsymbol{q},\omega) \equiv \bar{\chi}_0(\omega/\|\boldsymbol{q}\|)$, with $\bar{\chi}_0(x)$ the Lindhard function. The Lindhard function employed here and by HKM (1994) and HKMS (1994) is evaluated under the assumption that $\Lambda \downarrow 0$, taking into account the requirement $\lambda \ll \Lambda$; in this limit, a *sum* over FS 'patches' can be replaced by an *integral* (see Eq. (28) below). HKM (1994) consider the case of a short-range potential, replacing $v(\boldsymbol{q})$ by f_0 which they assume to be small, thus justifying their use of $f_0/[1+f_0\bar{\chi}_0(x)] \approx f_0 - f_0^2\bar{\chi}_0(x)$; $N(0)f_0$, with $N(0) \equiv k_F/[2\pi]$ the density of states at the Fermi energy, may be considered as the $\ell = 0$ component of the symmetric or charge part of the Landau parameter F^s_ℓ. HKMS (1994), on the other hand, consider the cases of the long-range Coulomb potential and super long-range potentials $v(\boldsymbol{q}) = g/\|\boldsymbol{q}\|^\gamma$ with g the coupling constant of interaction; in $d = 2$, the former corresponds to $\gamma = 1$ and the latter to $\gamma > 1$ and in the particular case of a logarithmic interaction to $\gamma = 2$. In these cases, corresponding to $\gamma = 1, 2$, taking into account the diverging behaviour of $v(\boldsymbol{q})$ as $\boldsymbol{q} \to \mathbf{o}$, these authors employ [10] $v(\boldsymbol{q})/[1+v(\boldsymbol{q})\bar{\chi}_0(x)] \approx 1/\bar{\chi}_0(x)$.

The second expansion employed by HKM (1994) and HKMS (1994) consists in that of the second exponential function on the RHS of Eq. (3) which these authors carry out to linear order (see footnote 7); this liner expansion

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[8] Here we are relying on a theorem which asserts uniqueness of coefficients in an asymptotic series expansion of a function with respect to a given asymptotic sequence (see Lauwerier 1977).

[9] Our cautious remark reflects our experience in dealing with the problem at hand where the most innocuous approximations turn out to change even the qualitative aspects of the results (Farid 1999b).

[10] The limit $v(q) \to \infty$ must be taken *after* evaluation of a pertinent integral over ω', for otherwise, in consequence of $1/\bar{\chi}_0(x) \sim -2x^2/N(0)$, for $|x| \to \infty$, the expression for $\delta G(k_\parallel,\omega)$ will involve a contribution proportional to $\int_{-\lambda_\parallel/2}^{\lambda_\parallel/2}\mathrm{d}q_\parallel \int_{-\lambda_\perp/2}^{\lambda_\perp/2}\mathrm{d}q_\perp\,[1/N(0)]/\|q\|^2$ (due to semi-circle contours at infinity in the complex ω'-plane) which is spurious. In the work by HKMS (1994) this contribution has been inadvertently (but correctly) neglected.

has been employed by the authors subsequent to their use of the pertinent expansion which we have indicated in the previous paragraph. For simplicity, but without loss of generality, here we only explicitly deal with the result corresponding to the case in which $v(\boldsymbol{q})$ is replaced by f_0. For the leading contribution (as specified by the lowest relevant power of f_0) to the *imaginary* part of $\delta G_{\rm f}(k_\parallel,\omega;S)$, HKM (1994) obtain (for $d=2$; below we suppress S in order to conform with the notation by HKM) [11]

$$\mathrm{Im}\delta G_{\rm f}(k_\parallel,\omega) = G_0(k_\parallel,\omega)G_0(k_\parallel,\omega)\mathcal{G}(k_\parallel,\omega), \tag{9}$$

$$\mathcal{G}(k_\parallel,\omega) := \frac{1}{2}\,\frac{f_0^2 N(0)}{(2\pi)^2}\,\mathrm{sgn}(\omega)\,\Big\{[\omega^2+(\omega-k_\parallel)^2/4]\ln\frac{|\omega-k_\parallel|}{\lambda_\perp} + [\omega^2-(\omega-k_\parallel)^2/4]\ln\frac{|\omega+k_\parallel|}{\lambda_\perp} - \frac{1}{2}\omega(2\omega-k_\parallel)\Big\}, \tag{10}$$

where $\lambda_\perp$ denotes a cut-off parameter restricting momentum integrations in the direction perpendicular to $\boldsymbol{n}_S$ to the interval $[-\lambda_\perp/2,\lambda_\perp/2]$. By neglecting $\Sigma(k,\omega)$ in the denominator of the expression on the RHS of Eq. (1), HKM (1994) identify $\mathcal{G}(k_\parallel,\omega)$ with $\mathrm{Im}\Sigma_{\rm f}(k_\parallel,\omega)$ to second order in f_0.

B. Some noteworthy observations

In the work by HKM (1994) as well as that by HKMS (1994), the authors apparently identify the zero of energy ω with the Fermi energy of the *non*-interacting system, i.e. $\omega_F^0=0$. In this sense the Green function G_0 on the RHS of Eq. (9) is in principle to be distinguished from G_0 as employed in our above considerations; HKM mention namely: "The location of the [quasi-particle] pole has been shifted from its bare value to $\omega = v_F' k_\parallel$ due to renormalization of the Fermi velocity ...". The authors thus fix the Fermi energy of the interacting system by replacing $k_\parallel$ in Eq. (10) by ω/v_F', where $v_F' = 1+F_0^s(1-F_0^s)\Lambda/[2\pi k_F]$. Subsequently, making use of $1/v_F' \approx 1-F_0^s\Lambda/[2\pi k_F]$, HKM (1994) obtain the imaginary part of the on-the-mass-shell self-energy, denoted by $\mathrm{Im}\Sigma_{\rm f}^{(2)}(\omega)|_{\rm pole}$, which is of the form $\mathrm{sgn}(\omega)\{A\omega^2\ln|\omega|+B\omega^2\}$; for the explicit forms of A and B see Eq. (63) in (HKM 1994). As we shall now demonstrate, the finding with regard to the quasi-particle pole having been shifted is based on an incorrect observation. To this end we first point out that $\mathrm{Im}\Sigma_{\rm f}^{(2)}(k_\parallel,\omega)$ stripped off of $\mathrm{sgn}(\omega)$ is zero at $\omega=0$, irrespective of the value chosen for $k_\parallel$, and moreover is negative elsewhere. The change of sign by $\mathrm{Im}\Sigma_{\rm f}^{(2)}(k_\parallel,\omega)$ is therefore entirely due to $\mathrm{sgn}(\omega)$. Stability of the system implies (and this is at the same time a corollary to a theorem due to Luttinger (1960) — see also Luttinger and Ward (1960)) that $\omega=0$ *must* be the Fermi energy of the interacting system. From this it follows that $\mathrm{Re}\Sigma_{\rm f}(0,0)=0$, which in turn implies that the self-energy as calculated by HKM (1994) (and by HKMS (1994)) *cannot* correspond to a conventional FL (see text following Eq. (1) above). In fact, as we shall demonstrate in this work, within the Haldane (1992) bosonisation approach, under *no* circumstance the system under consideration can be a conventional FL; neither can it be a LL, *even* for super-long-range interactions (see Section IV.B).

The property $\mathrm{Re}\Sigma_{\rm f}(0,0)=0 \iff \Sigma(0)=0$ enables us to identify $G_0(k_\parallel,\omega)$ in Eq. (9) with that employed by us at the outset of this work. Further, in consequence of $\Sigma(0)=0$, the appropriate expression for $\delta G(\omega)$ is [12]

$$\delta G(\omega) \sim \frac{\beta_0/(1-\beta_0)}{\omega} - i\,\frac{\alpha_0(1+\beta_0)}{1-\beta_0}\,\mathrm{sgn}(\omega)\,\phi(\omega), \tag{11}$$

rather than that in Eq. (2). Although the leading contributions in Eqs. (2) and (11) both scale like $1/\omega$, it must be noted that the coefficient of $1/\omega$ in Eq. (2), namely $1/(1-\beta_0)$, is positive while that in Eq. (11), i.e. $\beta_0/(1-\beta_0)$, is negative. In view of remarks following Eq. (7) above, we conclude that *if* the system under consideration is a FL, even though an unconventional one, then we have $\Xi = Z_{\boldsymbol{k}_F}$.

We have examined $\mathrm{Re}\delta G(\omega)$ — calculated independently, employing the same approximate treatment of Eq. (3) as employed by HKM (1994) and HKMS (1994) in their calculation of $\mathrm{Im}\delta G(\omega)$ — pertaining to the model under

[11] With reference to our remark in footnote 7, the RHS of this expression would have been multiplied by Ξ if our definition for $\delta G(\omega)$ had been employed.

[12] Neglect of $\Sigma(\omega)$ in the denominator of Eq. (1), as done by HKM (1994) and HKMS (1994), implies the following relation, to be compared with that in Eq. (11): $\delta G(\omega) \sim \beta_0/\omega - i\alpha_0\,\mathrm{sgn}(\omega)\,\phi(\omega)$, $\omega\to 0$.

consideration in considerable detail (results to be published shortly — Farid 1999b) and found that the second term on the RHS of Eq. (6) has *no* contribution to $\mathrm{Re}\delta G(\omega)$, for $\omega \to 0$, scaling like $1/\omega$ (see Eqs. (2) and (11) above). Our rigorous calculations show that contribution of this term to $\mathrm{Re}\delta G(\omega)$ scales like $\ln^2|\omega|$ for $\omega \to 0$; this result concerns the case where $v(\boldsymbol{q}) \to \infty$ — only for this case have we been able to perform our entire calculations fully analytically —, however numerically-obtained results corresponding to a general $v(\boldsymbol{q})$, including super long-range interactions, reveal a similar behaviour. With reference to our remarks following Eq. (7) above, we can therefore conclude that provided the first-order expansion of the exponential function within braces on the RHS Eq. (6) suffice (which does not seem to be the case, at least not for situations where the particle-particle interaction is super-long-range — see Section IV), the system under consideration must be an unconventional FL with $\Sigma(0) = 0$ and $Z_{\boldsymbol{k}_F} = \Xi$.

Two comments are in order here. First, if $\Sigma(0) \neq 0$ *and* $\Sigma(\omega)$ in the denominator of Eq. (1) were to be neglected (both of these conditions are implicit in the works by HKM (1994) and HKMS (1994)), then the leading asymptotic contribution to $\delta G(\omega)$ would diverge like $1/\omega^2$ for $\omega \to 0$ (see footnote 4). This would imply, since $\delta G(\omega)$ would not be integrable (in the Riemann sense) in a neighbourhood of $\omega = 0$, that *no* spectral representation for $\delta G(\omega)$ would exist (Farid 1999b), contradicting an obvious fact. Second, if $\Sigma(0) = 0$ *and* $\Sigma(\omega)$ in the denominator of Eq. (1) were to be neglected, then Eq. (1) would imply $\Sigma(\omega) \sim \omega^2 \delta G(\omega)$. From the spectral representation for $\delta G(\omega)$, on the other hand, it follows that to $\mathrm{Im}\delta G(\omega) \sim A\,\mathrm{sgn}(\omega)\,\ln|\omega|$ corresponds $\mathrm{Re}\delta G(\omega) \sim C$, for $\omega \to 0$, with C a constant (which may or may not be vanishing). Thus $\Sigma(\omega) \sim \omega^2 \delta G(\omega)$ would imply not only $\mathrm{Im}\Sigma(\omega) \sim A\,\mathrm{sgn}(\omega)\,\omega^2\,\ln|\omega|$, but also $\mathrm{Re}\Sigma(\omega) \sim C\,\omega^2$, for $\omega \to 0$. Now whereas $\mathrm{Im}\Sigma(\omega) \sim A\,\mathrm{sgn}(\omega)\,\omega^2\,\ln|\omega|$ would lead one to consider the metallic state at issue to be a FL, this would be at odds with $\mathrm{Re}\Sigma(\omega) \sim C\,\omega^2$, for, as we have mentioned in the paragraph preceding Eq. (1) above, in the asymptotic series of $\mathrm{Re}\Sigma(\omega)$, for $\omega \to 0$, pertaining to *interacting* FLs (both conventional and unconventional FLs), there *must* occur a term linear in [13] ω. These two comments make evident that in deducing $\Sigma(\omega)$ from $\delta G(\omega)$, in particular in the cases where one has to do with FLs, the self-energy in the denominator of Eq. (1) must not be neglected.

As our last observation in this Section, we mention that since $\mathrm{Im}\delta G(\omega) \sim [-f_0^2 N(0)/(2\pi)^2]\,\mathrm{sgn}(\omega)\ln|\omega|$ has been derived from the first-order expansion of the exponential function on the RHS of Eq. (3) (see footnote 7), taking into account that apparently $\Sigma(0) = 0$ — a property which, though not exclusively, is specific to LLs —, it is tempting to assume that higher-order contributions that correspond to the higher-order terms in the expansion of the mentioned exponential function involve appropriate powers of this first-order result, [14] thus allowing one to exponentiate the indicated first-order result and obtain a power-law behaviour for the self-energy. Explicitly, by pursuing this temptation, one obtains $\Sigma(\omega) \sim \beta_0[\cot(\pi\gamma_0) + i]\,\mathrm{sgn}(\omega)|\omega|^{1-2\gamma_0}$, for $\omega \to 0$, where [15]

$$\gamma_0 = \frac{1}{2}\big[1 - \frac{1}{(2\pi)^2} f_0^2 N(0)\big], \qquad \beta_0 = \frac{-1}{2(2\pi)^2}\, f_0^2 N(0)\, [\cot(\pi\gamma_0) + 1]\,. \tag{12}$$

The above asymptotic expression for $\Sigma(\omega)$ is that of LLs, with $\alpha \equiv 2\gamma_0$ the anomalous dimension (see, e.g., Farid 1999a). As it will become evident, this result is not tenable. However, the prospect that this result offers, that a LL metallic state would be feasible in $d = 2$, has been a further impetus for us to investigate the problem at hand in more detail, whence our exact treatment in Section IV.

IV. THE LONG-DISTANCE BEHAVIOUR OF THE SINGLE-PARTICLE GREEN FUNCTION; AN EXACT TREATMENT

Exact numerical evaluation of $\Sigma(k,\omega)$ within the Haldane (1992) bosonisation framework is in principle feasible for $d = 2$ but involves a considerable amount of investment in computational efforts as well as computational resources.

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[13] According to Eq. (11), which holds for FLs with $\Sigma(0) = 0$, $\omega^2\delta G(\omega) \sim [\beta_0/(1-\beta_0)]\omega$. Since we have shown $\Xi = Z_{k_F} \equiv 1/(1-\beta_0)$ under *all* circumstances as regards nature of $v(q)$, vanishing of this linear term would amount to $Z_{k_F} = 1$, or $\Xi = 1$ which through Eq. (4) would imply $\mathcal{F}^{-1}_{x=0,t=0}[\delta G_{\mathrm{b}}] = 0$. Evidently, this can occur *only* if $v(q) \equiv 0$.

[14] A cursory analysis suggests this to be the case. However, close inspection of this procedure reveals that a rigorous justification for exponentiating the above first-order result is far from trivial and, as we shall indirectly demonstrate in this work, incorrect. We note in passing that it *appears* to us that Bares and Wen (1993) employ a similar procedure in dealing with the low-energy behaviour of the single-particle Green function (see the paragraph following Eq. (42) in Bares and Wen 1993). If this indeed is the case, then in our opinion the conclusion arrived at by Bares and Wen (1993) may be of questionable nature.

[15] Using our definition for δG_{f}, in the following two expressions f_0^2 need be replaced by Ξf_0^2.

For our purpose, which at present in the main consists of establishing whether the metallic state of the model under consideration is a FL or otherwise, it is sufficient to determine the leading term in the large-$\|\boldsymbol{x}\|$ asymptotic expansion of the single-particle Green function in the $\boldsymbol{x}, t$-representation for t infinitesimally negative, i.e. $G(\boldsymbol{x}, t = 0^-)$, which can be achieved fully analytically.

A. General considerations

As can be seen from Eq. (3), for determination of the leading large-$\|\boldsymbol{x}\|$ term in the asymptotic series of $G(\boldsymbol{x}, t = 0^-)$ it is sufficient to calculate $\mathrm{Re}\big(2\pi i \Omega^{-2} \mathcal{F}^{-1}_{\boldsymbol{x}, t=0^-}[\delta G_{\mathrm{b}}]\big)$ for large $\|\boldsymbol{x}\|$. Before proceeding with this, we present some results which will set the stage for what follows. First, in $d = 1$ we have (see, e.g., Farid 1999b and Eq. (20) below)

$$-iG(x, t = 0^-) \sim \frac{Z_{k_F}}{\pi} \frac{\sin(k_F x)}{x} \quad \text{for } |x| \to \infty \quad \text{(FLs, } d = 1\text{)}, \tag{13}$$

$$-iG(x, t = 0^-) \sim \big[-2i\alpha C \int_0^{\Delta} \frac{\mathrm{d}k}{2\pi}\, k^{\alpha - 1} \cos(k) \big] \frac{\mathrm{sgn}(x)\, \exp(ik_F x)}{|x|^{1+\alpha}}, \text{ for } |x| \to \infty \text{ (LLs, } d = 1\text{)}. \tag{14}$$

We note that the asymptotic result pertaining to the LL in $d = 1$ can be directly inferred from the explicit expression for $G(x, t)$ pertaining to the one-dimensional Luttinger model (Luttinger 1963, Mattis and Lieb 1965) which is available (see, e.g., Voit 1994). For the calculation of the result in Eq. (14) we have *not* made use of this exact result, but employed the relationship between $-iG(x, t = 0^-)$ and the momentum distribution function $\mathsf{n}(k)$ (see Eq. (16) below). In Eq. (14), $\alpha \equiv 2\gamma_0$, with $0 < \alpha < 1$, is the anomalous dimension (see Eq. (12)), C a constant which features in the asymptotic series expansion of the momentum-distribution function $\mathsf{n}(k)$ for $k \to k_F$, *viz.* $\mathsf{n}(k) \sim 1/2 - C\,\mathrm{sgn}(k - k_F)\,|k - k_F|^{\alpha}$. Further, $\Delta > 0$ denotes a cut-off parameter whose value is on the order of the width of the derivative with respect to k of $\mathsf{n}(k)$ in the neighbourhood of $k = k_F$; the precise value of Δ is of no relevance to our work and it can be entirely disposed of if one employs the complete description of $\mathsf{n}(k)$ rather than its leading asymptotic terms for $k \to k_F$. That the RHS of Eq. (14), in contrast to that of Eq. (13), is complex-valued, has its origin in the fact that the expression in Eq. (14) concerns only the branch of the right-movers and therefore is determined by a single Fermi point, namely that at $k = k_F$.

For FLs in $d = 2$ one has (see, e.g., Farid 1999b)

$$-iG(\boldsymbol{x}, t = 0^-) \sim \frac{Z_{\boldsymbol{k}_F} \|\boldsymbol{k}_F\|^{1/2}}{2^{1/2}\pi^{3/2}} \frac{\sin\big(\|\boldsymbol{k}_F\|\,\|\boldsymbol{x}\| - \pi/4\big)}{\|\boldsymbol{x}\|^{3/2}}, \quad \text{for } \|\boldsymbol{x}\| \to \infty \quad \text{(FLs, } d = 2\text{)}, \tag{15}$$

while, in analogy with the expressions in Eqs. (13) and (14), for LLs in $d = 2$ one has a similar expression as in Eq. (15), with the power of $\|\boldsymbol{x}\|$, however, changed from $3/2$ to $3/2 + \alpha$ with $0 < \alpha < 1$. We point out that the power $3/2$ of $\|\boldsymbol{x}\|$ in $d = 2$, both for FLs and LLs, is composed of 1 and $1/2$, of which 1 (as well as α in the case of LLs) is the contribution from the radial part of a two-dimensional integral carried out in the cylindrical coordinate system and $1/2$ from the angular part.

For our further investigations it is important to understand the underlying reason for the qualitative difference in the long-distance behaviour of $-iG(\boldsymbol{x}, t = 0^-)$ pertaining to FLs and LLs. Here we only briefly consider this subject and for a detailed discussion refer the reader to our forthcoming publication (Farid 1999b). First, we point out that

$$-iG(\boldsymbol{x}, t = 0^-) = \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d}\, \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathsf{n}(\boldsymbol{k}). \tag{16}$$

With E_M the total energy of the interacting M-particle ground state, one can show that $\mu_N{:=}E_N - E_{N-1}$, with N the actual number of particles in the system, is equal to ω_F and that $\mu_N < \mu_{N+1}$, where $\mu_{N+1}{:=}E_{N+1} - E_N$. The latter inequality holds also in the thermodynamic limit, even though in this limit $\mu_{N+1} - \mu_N$ is infinitesimally small in the case of metals. Owing to this property, we can introduce μ, 'chemical potential', satisfying $\mu_N < \mu < \mu_{N+1}$. Suppose $G(\boldsymbol{k}, \omega)$ be *unbounded* at $\omega_{\boldsymbol{k}}$ ($\omega_{\boldsymbol{k}}$ may or may not be a pole, i.e. an isolated singularity) and let $\mathcal{C}_{\boldsymbol{k}}$ denote a circle of infinitesimally small radius in the complex energy plane (z-plane) centred around $\omega_{\boldsymbol{k}}$. In view of association of $\mathsf{n}(\boldsymbol{k})$ with $-iG(\boldsymbol{x}, t = 0^-)$, only those $\omega_{\boldsymbol{k}}$ are relevant to $\mathsf{n}(\boldsymbol{k})$ which satisfy $\omega_{\boldsymbol{k}} \leq \omega_F < \mu$. One has

$$\mathsf{n}(\boldsymbol{k}) = \Theta(\mu - \omega_{\boldsymbol{k}}) \int_{\mathcal{C}_{\boldsymbol{k}}} \frac{\mathrm{d}z}{2\pi i}\, G(\boldsymbol{k}, z) + \wp \int_{-\infty}^{\mu} \frac{\mathrm{d}\omega}{\pi}\, \mathrm{Im}[G(\boldsymbol{k}, \omega)]. \tag{17}$$

The energy $\omega_{\boldsymbol{k}}$ must satisfy $\omega_{\boldsymbol{k}} = \omega^0_{\boldsymbol{k}} + \Sigma(\boldsymbol{k}, \omega_{\boldsymbol{k}})$, the quasi-particle equation. This equation *may or may not* have a solution; however, ω_F is *always* a solution of this equation (see Farid 1999a). If $\omega_{\boldsymbol{k}}$ satisfies this equation and, moreover, $\Sigma(\boldsymbol{k}, \omega)$ is a continuously-differentiable function of ω in a neighbourhood of $\omega = \omega_{\boldsymbol{k}}$, then one has

$$\int_{\mathcal{C}_{\boldsymbol{k}}} \frac{\mathrm{d}z}{2\pi i}\, G(\boldsymbol{k}, z) = Z_{\boldsymbol{k}}, \tag{18}$$

where

$$Z_{\boldsymbol{k}} := \Big(1 - \frac{\partial \Sigma(\boldsymbol{k}, \omega)}{\partial \omega}\Big|_{\omega=\omega_{\boldsymbol{k}}}\Big)^{-1}. \tag{19}$$

A specific aspect of FLs, in comparison with LLs and marginal FLs, is that $\Sigma(\boldsymbol{k}_F, \omega)$ pertaining to FLs is a continuously differentiable function of ω in a neighbourhood of $\omega = \omega_F$ (see text preceding Eq. (1) above). As will become evident, $Z_{\boldsymbol{k}_F} \in (0, 1]$, which applies to FLs (specifically, but not exclusively) is of special significance (see Eqs. (13), (15) and (31)). Amongst others, by continuity, $Z_{\boldsymbol{k}_F} \neq 0$ implies a non-vanishing $Z_{\boldsymbol{k}}$ in a neighbourhood of $\boldsymbol{k} = \boldsymbol{k}_F$ which together with $\Theta(\mu - \omega_{\boldsymbol{k}})$ result in the fact that the first term on the RHS of Eq. (17) is a non-trivial function which is non-vanishing only over a *finite* interval. To appreciate the far-reaching consequence of this property in the context of our present discussions, let us consider a FL in $d = 1$ for which we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx} \left\{\Theta(\mu - \omega_k) \int_{\mathcal{C}_k} \frac{\mathrm{d}z}{2\pi i}\, G(k, z)\right\} &= \int_{-k_F}^{k_F} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx}\, Z_k \\ &= \frac{Z_{k_F}}{\pi}\, \frac{\sin(k_F x)}{x} - \frac{1}{ix} \int_{-k_F}^{k_F} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx}\, \frac{\mathrm{d}}{\mathrm{d}k} Z_k,\end{aligned} \tag{20}$$

where the second line on the RHS of Eq. (20) has been obtained through integrating by parts. Under the assumption of integrability of $\mathrm{d}Z_k/\mathrm{d}k$ over $[-k_F, k_F]$, it can be readily demonstrated that for $|x| \to \infty$ the second term on the RHS of Eq. (20) is asymptotically sub-dominant with respect to the first term, whence absence of any contribution due to this second term to the leading asymptotic term of $-iG(x, t = 0^-)$, $|x| \to \infty$, as presented in Eq. (13). One can further demonstrate that the unbounded support of the second term on the RHS of Eq. (17) results in the fact that the contribution of this term to the leading term in the large-$|x|$ asymptotic series of $-iG(x, t = 0^-)$ is vanishing. This simple example brings out the importance of a contribution of *finite* support to $\mathsf{n}(\boldsymbol{k})$.

If one applies the same procedure to a case (say, in $d = 1$) where the first term on the RHS of Eq. (17) is identically vanishing (as is the case in the one-dimensional Luttinger model), then one readily establishes that the leading contribution to $-iG(x, t = 0^-)$, for $|x| \to \infty$, decays faster than $1/x$ for $|x| \to \infty$; the expression in Eq. (14) is an evidence to this statement. On the other hand, if Z_k is non-vanishing but $\mathrm{d}Z_k/\mathrm{d}k$ is non-integrable over $[-k_F, k_F]$ — a hypothetical situation —, then integration by parts, as applied in Eq. (20), cannot be employed and an analysis based on some specific aspects of Z_k reveals that $-iG(x, t = 0^-) \sim o(1)$, for $|x| \to \infty$, where $o(1)$ signifies a vanishingly small function which in the case at hand is asymptotically more dominant in comparison with $1/x$.

In dealing with the asymptotic behaviour of $-iG(x, t = 0^-)$ for $|x| \to \infty$ in the case of the one-dimensional Luttinger model, one has to bear in mind the important fact that here *all* functions of momentum are defined over $(-\infty, \infty)$ and that a mathematically sound treatment of the model requires introduction of appropriate cut-offs (which in considerations concerning asymptotic behaviour of correlation functions at large distance must be 'soft' cut-offs, as opposed to 'sharp' ones) in dealing with functions of momentum (Mattis and Lieb 1965). As a consequence of this, the boundary contributions obtained through integration by parts as applied to the second term on the RHS of Eq. (17) (*c.f.* the first term on the RHS of Eq. (20)) are vanishing. It is for this very reason that a term decaying like $1/x$ is missing in the expression on the RHS of Eq. (14); this missing term, which we denote by $-iG(x, t = 0^-)|_{\text{missing}}$, would have the following functional form

$$\begin{aligned}-iG(x, t = 0^-)\big|_{\text{missing}} &\equiv \frac{1}{2\pi i}\, \frac{\exp(i\tilde{\lambda}x/2)\mathsf{n}(\tilde{\lambda}/2) - \exp(-i\tilde{\lambda}x/2)\mathsf{n}(-\tilde{\lambda}/2)}{x} \\ &\sim \frac{-1}{2\pi i}\, \frac{\exp(-i\tilde{\lambda}x/2)}{x},\end{aligned} \tag{21}$$

where in the second expression we have employed $\mathsf{n}(-\tilde{\lambda}/2) \sim 1$ (due to the assumed absence of a momentum cut-off) and $\mathsf{n}(\tilde{\lambda}/2) \sim 0$ which apply for sufficiently large (positive) $\tilde{\lambda}$. Thus without proper treatment of functions

of momentum, the distinction between FLs and LLs (here in $d = 1$) would be obliterated in as far as the decaying behaviour of $-iG(x, t = 0^-)$ for large $|x|$ is concerned. The same statement applies for arbitrary d. Below, by explicitly analysing the large-$\|\boldsymbol{x}\|$ behaviour of $-iG(\boldsymbol{x}, t = 0^-)$ for the model under consideration, we demonstrate that within the Haldane's (1992) bosonisation scheme for $d > 1$, the large-$\|\boldsymbol{x}\|$ behaviour of $-iG(\boldsymbol{x}, t = 0^-)$ *cannot* involve an anomalous dimension similar to the case in $d = 1$, that is irrespective of whether the system under consideration be a FL or otherwise, the large-$\|\boldsymbol{x}\|$ behaviour of $-iG(\boldsymbol{x}, t = 0^-)$ is always FL-like. This suggests an improper treatment of $\mathsf{n}(\boldsymbol{k})$ within this scheme when $d > 1$. We establish this fact in Section V.

B. The long-distance behaviour of the Green function within the framework of the Haldane bosonisation

It can be readily shown that

$$\begin{aligned}\mathrm{Re}\left[\frac{2\pi i}{\Omega^2}\mathcal{F}^{-1}_{\boldsymbol{x},t=0^-}[\delta G_{\mathrm{b}}]\right] = \frac{-2}{(2\pi)^3}\Big\{ \int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\int_0^{Q(\varphi)}\mathrm{d}\|\boldsymbol{q}\|\ \cos\big(\|\boldsymbol{q}\|\,\|\boldsymbol{x}\|\cos(\varphi-\Phi_{\hat{\boldsymbol{x}}})\big)\\ \times\wp\int_0^1\frac{\mathrm{d}\omega}{\pi}\,\frac{\mathrm{Im}[W(\boldsymbol{q},-\omega)]}{\big(\omega+\cos(\varphi)\big)^2}\\ +\int_{\pi/2}^{3\pi/2}\frac{\mathrm{d}\varphi}{2\pi}\int_0^{Q(\varphi)}\mathrm{d}\|\boldsymbol{q}\|\ \cos\big(\|\boldsymbol{q}\|\,\|\boldsymbol{x}\|\cos(\varphi-\Phi_{\hat{\boldsymbol{x}}})\big)\\ \times\mathrm{Re}[W^{(0,1)}\big(\|\boldsymbol{q}\|,\cos(\varphi)\big)]\Big\},\end{aligned} \tag{22}$$

where $\Phi_{\hat{\boldsymbol{x}}}$ is the planar angle between $\boldsymbol{x}$ and $\boldsymbol{n}_S$ and $Q(\varphi)$ specifies the boundary of the rectangular area $[-\lambda_\perp/2, \lambda_\perp/2] \times [-\lambda_\|/2, \lambda_\|/2]$ in the momentum space over which momentum integrations are carried out (for the origins of these parameters see HKM (1994) and HKMS (1994)). In general [16] $\lambda_\perp \ll \lambda_\|$. With $\Phi_0{:=}\arctan(\lambda_\perp/\lambda_\|)$, we have $Q(\varphi) = \lambda_\|/[2\cos(\varphi)]$, for $0 \le \varphi < \Phi_0$ and $2\pi - \Phi_0 \le \varphi < 2\pi$; $Q(\varphi) = \lambda_\perp/[2\sin(\varphi)]$, for $\Phi_0 \le \varphi < \pi - \Phi_0$; $Q(\varphi) = -\lambda_\|/[2\cos(\varphi)]$, for $\pi - \Phi_0 \le \varphi < \pi + \Phi_0$, and $Q(\varphi) = -\lambda_\perp/[2\sin(\varphi)]$, for $\pi + \Phi_0 \le \varphi < 2\pi - \Phi_0$. Further, $W^{(0,1)}(\boldsymbol{q},\omega){:=}\mathrm{d}W(\boldsymbol{q},\omega)/\mathrm{d}\omega$. After an appropriate regularisation (Farid 1999b), in order to rendering it meaningful, it can be rigorously shown that

$$\wp\int_0^1\frac{\mathrm{d}\omega}{\pi}\,\frac{\mathrm{Im}[W(\boldsymbol{q},-\omega)]}{\big(\omega+\cos(\varphi)\big)^2} \sim \frac{-N(0)v^2(\boldsymbol{q})}{\pi\big(1+v(\boldsymbol{q})N(0)\big)^2}\,\ln|\cos(\varphi)|, \quad \varphi\to\pi/2, 3\pi/2; \tag{23}$$

that is to say, for $\varphi \in [0, 2\pi]$ the most divergent contribution of the integral on the left-hand side (LHS) arises from neighbourhoods of $\varphi = \pi/2$ and $\varphi = 3\pi/2$. Further,

$$\mathrm{Re}\big[W^{(0,1)}(\|\boldsymbol{q}\|,\omega)\big] = \frac{-2\big(1+v(\boldsymbol{q})N(0)\big)v^3(\boldsymbol{q})N^2(0)}{d^2+e^2}\,\omega, \quad |\omega| \le 1, \tag{24}$$

where

$$\begin{aligned} d{:=}&\big(1+v(\boldsymbol{q})N(0)\big)^2 - \big[\big(1+v(\boldsymbol{q})N(0)\big)^2 + v^2(\boldsymbol{q})N^2(0)\big]\omega^2,\\ e{:=}&2\big(1+v(\boldsymbol{q})N(0)\big)v(\boldsymbol{q})N(0)|\omega|(1-\omega^2)^{1/2}.\end{aligned} \tag{25}$$

For $v(\boldsymbol{q}) \to \infty$ holds $\mathrm{Re}\big[W^{(0,1)}(\|\boldsymbol{q}\|,\omega)\big] = -2\omega/\big[N(0)\big(1+4\omega^2(1-\omega^2)\big)\big]$. Evidently, $\mathrm{Re}\big[W^{(0,1)}(\|\boldsymbol{q}\|,\omega)\big]$ is a regular function of ω so that the second integral on the RHS of Eq. (22) is finite for all $\|\boldsymbol{x}\|$ and therefore from the view point our present considerations of no significance. Making use of the property $|\int_a^b \mathrm{d}x\, f(x)| \le \int_a^b \mathrm{d}x\,|f(x)|$, applicable to absolutely-integrable functions $f(x)$ over $[a, b]$, it can be readily shown that the RHS of Eq. (22) is bounded for *all*

[16] In both HKM (1994) and HKMS (1994) $\lambda_\perp$ is set equal to λ while $\lambda_\|$ is equated with $+\infty$. In (Kwon, Houghton and Marston 1995 — see text following Eq. (27) herein) conditions under which $\lambda_\perp \ll \lambda_\|$ or $\lambda_\perp = \lambda_\| = \lambda$, etc., apply have been explicitly indicated (it appears, however, that "perpendicular directions" in this text should be understood as meaning "parallel directions"). The precise relationship between $\lambda_\perp$ and $\lambda_\|$ is of *no* consequence to our observations to be made further on.

values of $\|\boldsymbol{x}\|$. As a matter of fact, explicit analytic calculation (Farid 1999b) shows that for large $\|\boldsymbol{x}\|$ both terms on the RHS of Eq. (22) approach zero. For completeness, for the system under consideration to be a LL, it is necessary that for $\Phi_{\hat{\boldsymbol{x}}} = 0$ the RHS of Eq. (22) behave like $-\alpha \ln \|\boldsymbol{x}\|$ for large $\|\boldsymbol{x}\|$, with $\alpha \in (0,1)$.

The above observations are interesting for the reason that they imply that *irrespective* of the nature of the interaction function $v(\boldsymbol{q})$, i.e. whether $v(\boldsymbol{q})$ be short-, long- or super-long range, the metallic state of the system would be a FL (in the narrow sense that $Z_{\boldsymbol{k}_F} \neq 0$ — see Farid 1999a). For short- and the long-range Coulomb interaction our observation would then agree with that by HKM (1994) and HKMS (1994); for super-long ranged interactions, however, we would be in conflict with both HKMS (1994) and Bares and Wen (1993) who obtain a LL state and a "$Z_{k_F} = 0$ Fermi liquid", respectively. We point out that work by Bares and Wen (1993) is based on the random-phase approximation (RPA) so that a discrepancy between our finding and that by the latter authors in not inconceivable (see however footnote 14).

We have analysed the work by HKMS (1994) and found that the reason for their just-mentioned finding lies, amongst others, in their use the following approximation (see Eq. (26) in HKMS (1994))

$$\bar{\chi}_0(x) = -\frac{N(0)}{2x^2} + i\eta + \mathcal{O}(1/x^4), \tag{26}$$

which for sufficiently small values of the coupling constant of interaction, i.e. g in $v(\boldsymbol{q}) = g/\|\boldsymbol{q}\|^\gamma$, induces spurious poles in, e.g., $W(\boldsymbol{q},\omega)$ over a substantial part of the $\boldsymbol{q}$-space. [17] Thus we seem to have no alternative but to accept the verdict stated at the beginning of the previous paragraph.

C. Investigating some crucial aspects

In view of the above conflicting findings, we now turn to investigating in some detail a number of crucial aspects in Haldane's (1992) bosonisation scheme. As the point of departure, we consider the case of non-interacting fermions and determine the behaviour of $G_0(\boldsymbol{x}, t = 0^-)$ as evaluated from the expression in Eq. (5). This will lead us to a key observation concerning a shortcoming of the operator algebra (see Eq. (39) below) that underlies Eq. (3). We have [18]

$$G_{0;\mathrm{f}}(\boldsymbol{x}, t = 0^-; S) = \frac{\Lambda}{(2\pi)^2} \frac{\exp\big(i\boldsymbol{k}_F(S)\cdot\boldsymbol{x}\big)}{\boldsymbol{x}\cdot\boldsymbol{n}_S - i\,\eta}, \quad \eta \downarrow 0. \tag{27}$$

We obtain the non-interacting Green function $G_0(\boldsymbol{x}, t = 0^-)$ through averaging (below indicated by 'Av') the RHS of Eq. (27) over all 'patches' of the FS (for some subtle aspects related to this see further on). In doing so, we make use of $\Lambda \ll 1$ and thus employ the following result which is applicable to the limit $\Lambda \downarrow 0$ (see text following Eq. (8) above),

$$\sum_S (\ldots) = \frac{\|\boldsymbol{k}_F\|}{\Lambda} \int_0^{2\pi} \mathrm{d}\varphi\, (\ldots). \tag{28}$$

With $\boldsymbol{k}_F(S) = \|\boldsymbol{k}_F\|\,\boldsymbol{n}_S$, where we assume $\|\boldsymbol{k}_F\|$ to be independent of patch index, and $\boldsymbol{x}\cdot\boldsymbol{n}_S = \|\boldsymbol{x}\|\cos(\varphi)$, we have

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[17] It can be straightforwardly shown that for $|\omega| \leq 1$, $\mathrm{Im}\big[W(q,\omega)\big] = -v^2(q)N(0)\,|\omega|\,(1-\omega^2)^{1/2}/(a-b\omega^2)$ where $a{:=}\big(1 + v(q)N(0)\big)^2$ and $b{:=}1{+}2v(q)N(0)$. Since $a/b > 1$, it is evident that $\mathrm{Im}\big[W(q,\omega)\big]$ is bounded for *all* $|\omega| \in [0,1]$. Employing the approximation in Eq. (26), on the other hand, results in $\mathrm{Im}\big[W(q,\omega)\big] = -(\pi/2)\,\mathrm{sgn}(\omega)\,\sqrt{v(q)N(0)/2}\,\delta\big(\omega - \mathrm{sgn}(\omega)\,\sqrt{v(q)N(0)/2}\big)$. This implies that $W(q,\omega)$ has simple poles at $\omega = \omega_\pm {:=} \pm\sqrt{v(q)N(0)/2}$ which for sufficiently large $\|q\|$, or sufficiently small g *and* over a substantial part of the relevant q-space, are located in the interval $(-1,1)$. With the plasma frequency ω_p obeying $\omega_\mathrm{p} = \big(g\|k_F\|/[2\pi]\big)^{1/2}$, it is interesting to mention that HKMS (1994) assert that "If $\omega_\mathrm{p} \gg \lambda$ the system does not see the plasmons and Fermi liquid behaviour is retained. In the opposite limit of $\omega_\mathrm{p} \ll \lambda$, however, the plasmons destroy the Fermi liquid." The property $\omega_\mathrm{p} \propto |\omega_\pm|$ leaves no doubt that in the case at hand destruction of the FL state far from being caused by plasmons is due to failure of the approximate expression in Eq. (26) to be qualitatively correct.

[18] It is readily verified that $\int_{-\infty}^{\infty}\mathrm{d}x\,\exp(-ikx)\int_{-\infty}^{\infty}\mathrm{d}t\,\exp(i\omega t)/\big[2\pi\big(x - t + i\,\eta\,\mathrm{sgn}(t)\big)\big] = 1/\big(\omega - k + i\,\eta\,\mathrm{sgn}(\omega)\big) \equiv G_0(k,\omega)$, where $\eta \downarrow 0$. Thus the additional pre-factor in Eq. (27), namely $\Lambda/(2\pi)$, is specific to the 'patch' Green function which, as we shall see, is crucial for an averaging procedure.

$$-iG_0^{\rm Av}(\boldsymbol{x},t=0^-)=\frac{\|\boldsymbol{k}_F\|}{\|\boldsymbol{x}\|}\int_0^{2\pi}\frac{{\rm d}\varphi}{2\pi}\,{\rm e}^{i\|\boldsymbol{k}_F\|\,\|\boldsymbol{x}\|\cos(\varphi)}\,\delta\big(\cos(\varphi)\big)$$
$$-\frac{i\,\|\boldsymbol{k}_F\|}{2\pi\|\boldsymbol{x}\|}\,\wp\int_0^{2\pi}\frac{{\rm d}\varphi}{2\pi}\,\frac{\exp\big(i\|\boldsymbol{k}_F\|\,\|\boldsymbol{x}\|\cos(\varphi)\big)}{\cos(\varphi)}, \tag{29}$$

which after some algebra results in

$$-iG_0^{\rm Av}(\boldsymbol{x},t=0^-)\sim\frac{\|\boldsymbol{k}_F\|}{2\pi\,\|\boldsymbol{x}\|}+\frac{\|\boldsymbol{k}_F\|^{1/2}}{2^{1/2}\pi^{3/2}}\,\frac{\sin(\|\boldsymbol{k}_F\|\,\|\boldsymbol{x}\|-\pi/4)}{\|\boldsymbol{x}\|^{3/2}},\quad \|\boldsymbol{x}\|\to\infty, \tag{30}$$

of which the second term on the RHS is obtained through application of the method of stationary phase (see, e.g., Ch. 4 in Murray 1974), taking into account that ${\rm d}\cos(\varphi)/{\rm d}\varphi\equiv-\sin(\varphi)=0$ for $\varphi=0,\pi,2\pi$ with $\varphi=0,2\pi$ coinciding with the end-points of the interval of integration (see pp. 76 and 77 in Murray 1974). Since a system of non-interacting fermions is a FL, this expression must be compared with that in Eq. (15).

The first term on the RHS of Eq. (30) is not unexpectedly spurious: it is a contribution from points $\varphi=\pi/2,3\pi/2$ at which for large $\|\boldsymbol{x}\|$ the condition for validity of Eq. (3) and therefore of Eq. (5), namely $|\boldsymbol{x}\cdot\boldsymbol{\tau}_S|\ll1/\Lambda$, is violated.

The second term on the RHS of Eq. (30), contrary to the first term, cannot be suffering from violation of condition $|\boldsymbol{x}\cdot\boldsymbol{\tau}_S|\ll1/\Lambda$ since it entirely consists of contributions from the turning points of $\cos(\varphi)$, namely $\varphi=0,\pi,2\pi$ (see above), and at these points we have $|\boldsymbol{x}\cdot\boldsymbol{\tau}_S|=0$ for *all* $\boldsymbol{x}$. Having established this fact, it follows that the perfect agreement between this term and the RHS of Eq. (15) with $Z_{\boldsymbol{k}_F}=1$ is not accidental.

For our following considerations we introduce $\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,\varphi),t}[\delta G_{\rm b}]$ as representing $\mathcal{F}^{-1}_{\boldsymbol{x},t}[\delta G_{\rm b}]$ in the cylindrical coordinates. Since $\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,\varphi),0^-}[\delta G_{\rm b}]\equiv\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,\varphi+m\pi),0^-}[\delta G_{\rm b}]$ with m any integer (Farid 1999b), it can be readily shown that

$$-iG^{\rm Av}(\boldsymbol{x},t=0^-)\sim\frac{\Xi\,\|\boldsymbol{k}_F\|}{2\pi\|\boldsymbol{x}\|}\,\exp\big(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,\pi/2),t=0^-}[\delta G_{\rm b}]\big)$$
$$+\frac{\Xi\,\|\boldsymbol{k}_F\|^{1/2}}{2^{1/2}\pi^{3/2}}\,\exp\big(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,0),t=0^-}[\delta G_{\rm b}]\big)$$
$$\times\frac{\sin\big(\|\boldsymbol{k}_F\|\,\|\boldsymbol{x}\|-\pi/4\big)}{\|\boldsymbol{x}\|^{3/2}},\quad \|\boldsymbol{x}\|\to\infty. \tag{31}$$

All our remarks concerning the nature of the two terms on the RHS of Eq. (30) apply to those on the RHS of Eq. (31).

Since, as we have mentioned above (see text following Eq. (25) above), $\exp\big(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{(\|\boldsymbol{x}\|,0),t=0^-}[\delta G_{\rm b}]\big)\to1$ for $\|\boldsymbol{x}\|\to\infty$, independent of the nature of the particle-particle interaction, it follows that we can directly compare the second term on the RHS of Eq. (31) with the term on the RHS of Eq. (15), and thus identify [19] Ξ with $Z_{\boldsymbol{k}_F}$. This simple observation leads us to draw a far-reaching conclusion: in light of our discussions following Eq. (7) and with reference to Eq. (11), $\Xi=Z_{\boldsymbol{k}_F}$ *and* $\Sigma(0)=0$. Moreover, our finding that the real part of the second term on the RHS of Eq. (6) does *not* scale like $1/\omega$, as $|\omega|\to0$ (see paragraph following Eq. (11) above), is *not* an artifact of the first-order expansion employed by us (although our result that this contribution scales like $\ln^2|\omega|$ may best be).

V. THE ALGEBRAIC STRUCTURE OF THE HALDANE BOSONISATION SCHEME REVISITED

In order to gain insight into the origin of the above results, we proceed with an investigation of the algebraic structure underlying Hadalne's (1992) bosonisation scheme which is embodied by a Kac-Moody algebra. In deriving this algebra for the charge-current operators $\hat{J}(S;\boldsymbol{q})$ and $\hat{J}(T;\boldsymbol{q})$ pertaining to FS 'patches' S and T,

$$\hat{J}(S;\boldsymbol{q}):=\sum_{\boldsymbol{k}\,\sigma}\Theta(S;\boldsymbol{k}+\boldsymbol{q})\Theta(S;\boldsymbol{k})\Big\{\hat{a}^\dagger_{\boldsymbol{k}+\boldsymbol{q}\,\sigma}\hat{a}_{\boldsymbol{k}\,\sigma}-\delta_{\boldsymbol{q},\mathbf{0}}\,\mathsf{n}(\boldsymbol{k})\Big\}, \tag{32}$$

[19] For completeness, if $\exp\big(2\pi i\Omega^{-2}\mathcal{F}^{-1}_{(\|x\|,0),t=0^-}[\delta G_{\rm b}]\big)\to C$ for $\|x\|\to\infty$, then we would have $C\Xi=Z_{k_F}$. If, on the other hand, $2\pi i\Omega^{-2}\mathcal{F}^{-1}_{(\|x\|,0),t=0^-}[\delta G_{\rm b}]\to\infty$ for $\|x\|\to\infty$, Eq. (31) *cannot* be compared with Eq. (15) and therefore Ξ *cannot* be identified with Z_{k_F}. Recall that (see text following Eq. (4) above) $\Xi\neq0$, for otherwise $G_{\rm f}\equiv0$.

one starts from the following *exact* result (below $[\,,\,]_-$ denotes commutation)

$$[\hat{J}(S;\boldsymbol{q}),\hat{J}(T;\boldsymbol{p})]_- = \sum_{\boldsymbol{k}\,\sigma}\Big\{\Theta(S;\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\Theta(S;\boldsymbol{k}+\boldsymbol{p})\,\Theta(T;\boldsymbol{k}+\boldsymbol{p})\,\Theta(T;\boldsymbol{k}) \\ -\Theta(S;\boldsymbol{k}+\boldsymbol{q})\,\Theta(S;\boldsymbol{k})\,\Theta(T;\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\Theta(T;\boldsymbol{k}+\boldsymbol{q})\Big\}\,\hat{a}^{\dagger}_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}\,\sigma}\hat{a}_{\boldsymbol{k}\,\sigma} \tag{33}$$

which is obtained through repeated application of the canonical anti-commutation relations that hold amongst the fermion creation and annihilation operators $\hat{a}^{\dagger}_{\boldsymbol{k}\,\sigma}$ and $\hat{a}_{\boldsymbol{k}\,\sigma}$, respectively; here σ denotes the spin index. Above $\Theta(S;\boldsymbol{k})$ stands for the characteristic function of the FS 'patch' S, i.e. it is equal to 1 when $\boldsymbol{k}$ lies inside the 'squat box' associated with 'patch' S and zero otherwise. *Two* fundamental assumptions (the second of which to our best knowledge has never earlier been acknowledged as such) are necessary in order to reduce the expression in Eq. (33) into a Kac-Moody algebra. The first of these amounts to a decoupling of scattering processes amongst different 'squat boxes' and is expressed through the following two equations:

$$\Theta(S;\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\Theta(T;\boldsymbol{k}+\boldsymbol{p}) = \delta_{S,T}\,\Theta(S;\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\Theta(S;\boldsymbol{k}+\boldsymbol{p}), \tag{34}$$

$$\Theta(S;\boldsymbol{k})\,\Theta(T;\boldsymbol{k}+\boldsymbol{q}) = \delta_{S,T}\,\Theta(S;\boldsymbol{k})\,\Theta(S;\boldsymbol{k}+\boldsymbol{q}). \tag{35}$$

In order to expose the second *approximation*, we first need to proceed through two intermediate steps. In the first, $\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{0}}\,\mathsf{n}(\boldsymbol{k})$ is added and subsequently subtracted from $\sum_\sigma \hat{a}^{\dagger}_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}\,\sigma}\hat{a}_{\boldsymbol{k}\,\sigma}$ in Eq. (33), taking into account the assumptions in Eqs. (34) and (35). In the second, on the basis of the inequalities $\lambda \ll \Lambda \ll \|\boldsymbol{k}_F\|$ an argument is made (see Houghton and Marston 1993 and Haldane 1992) to write the thus-obtained simplified expression as follows

$$[\hat{J}(S;\boldsymbol{q}),\hat{J}(T;\boldsymbol{p})]_- = \delta_{S,T}\Big\{\sum_{\boldsymbol{k}}\Theta(S;\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\Theta(S;\boldsymbol{k})\,\big[\Theta(S;\boldsymbol{k}+\boldsymbol{p})-\Theta(S;\boldsymbol{k}+\boldsymbol{q})\big] \\ \times\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{0}}\,\mathsf{n}(\boldsymbol{k}) + \text{some error term}\Big\}. \tag{36}$$

Owing to $\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{0}}$, $\boldsymbol{p}$ in the summand can be replaced by $-\boldsymbol{q}$, upon which, following a shift in the summation variable, the RHS of Eq. (36) in the thermodynamic limit transforms into an expression involving

$$g_S(\boldsymbol{q}) := \int_{\bar{S}} \mathrm{d}^d\boldsymbol{k}\,\big[\mathsf{n}(\boldsymbol{k}-\boldsymbol{q}/2)-\mathsf{n}(\boldsymbol{k}+\boldsymbol{q}/2)\big], \tag{37}$$

in which $\bar{S}$ denotes the 'squat box' defined by the region of the $\boldsymbol{k}$-space for which holds $\Theta(S;\boldsymbol{k}+\boldsymbol{q}/2)\Theta(S;\boldsymbol{k}-\boldsymbol{q}/2)=1$. Now as for the second approximation to which we have referred above, Haldane (1992) [20] asserts that the value of $g_S(\boldsymbol{q})$, for $\|\boldsymbol{q}\| \ll \lambda$, which is "the number of allowed k-space points inside the volume of reciprocal space swept out by displacing the patch of Fermi surface by $\boldsymbol{q}$", "is independent of the detailed structure of $\langle n_{\boldsymbol{k}}\rangle_0$ $[\equiv \mathsf{n}(\boldsymbol{k})]$ near the Fermi surface and only involves the change in the asymptotic values of the occupation factor from deep inside to far outside the Fermi surface." This statement, as we shall explicitly demonstrate below, is applicable *only* in $d=1$. Moreover, we unequivocally demonstrate that in $d>1$, specifically in $d=2$, the employed Kac-Moody algebra for current operators does *not* allow $Z_{\boldsymbol{k}_F}$ to be vanishing and the large-$\|\boldsymbol{x}\|$ behaviour of $-iG(\boldsymbol{x},t=0^-)$ to be different from that of FLs, in full conformity with our finding based on our explicit calculation of $-iG(\boldsymbol{x},t=0^-)$ for $\|\boldsymbol{x}\|\to\infty$.

It is on the basis of the view point of Haldane's, leading to

$$g_S(\boldsymbol{q}) = \Lambda^{d-1}\,\boldsymbol{q}\cdot\boldsymbol{n}_S, \tag{38}$$

that Eq. (36) upon suppression of 'some error term' furnishes the charge-current operators $\{\hat{J}(S;\boldsymbol{q})\}$ with the Kac-Moody current algebra (Brink and Hanneaux 1988, Goddard and Olive 1986)

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[20] See p. 20 in (Haldane 1992). The function $g_S(q)$ as defined in Eq. (37) is identical to $g_\alpha(q)$ as defined in Eq. (35) in (Haldane 1992), which, corrected for some misprints, reads: $g_\alpha(q)=\sum_k \theta_\alpha(k+q)\,\theta_\alpha(k)\,\langle(n_{k+q}-n_k)\rangle_0$. It is perhaps useful to mention that a and V in Eq. (36) of (Haldane 1992) are defined as $a{:=}\Lambda^{d-1}$ and $V{:=}2(L/[2\pi])^d$. Further, Λ and λ in the work by Haldane (1992) are to be identified with respectively λ and Λ in the work by Houghton and Marston (1993) whose notation we have adopted in our present work.

$$\Big[\hat{J}(S;\boldsymbol{q}),\hat{J}(T;\boldsymbol{p})\Big]_{-}=2\delta_{S,T}\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{o}}\,\Omega\,\boldsymbol{q}\cdot\boldsymbol{n}_S, \tag{39}$$

where $\Omega{:=}\Lambda^{d-1}(L/[2\pi])^d$, with L the length of the side of the d-dimensional macroscopic hyper-cube within which the system under consideration is confined (Λ is defined in the paragraph containing Eq. (3) above). For later reference, we mention that the RHS of Eq. (39) is known as *quantum anomaly*.

Now we establish that in a rigorous treatment of the problem at hand, 'some error term' in Eq. (36) and/or the 'inter-box' scattering events, as expressed in Eqs. (34) and (35), must play prominent roles. [21] In this connection we mention that the estimation as put forward by Houghton and Marston (1993) — see text following Eq. (3.5a) herein; the authors explicitly consider the case of $d=3$ —, namely that the magnitude of 'some error term' were by on the order of $1/\sqrt{\mathcal{N}}$ smaller than that of the quantum anomaly, [22] where $\mathcal{N}{:=}\|\boldsymbol{k}_F\|/\lambda$, aside from being based on the approximation in which $\big(\hat{a}^{\dagger}_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}\,\sigma}\hat{a}_{\boldsymbol{k}\,\sigma}-\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{o}}\,\mathsf{n}(\boldsymbol{k})\big)$ is replaced by the c-number $(1-\delta_{\boldsymbol{q}+\boldsymbol{p},\mathbf{o}})$, relies on the assumption of validity of the RHS of Eq. (39). Below we explicitly deal with cases $d=1$ and $d=2$ and demonstrate that the restrictions $\lambda\ll\Lambda\ll\|\boldsymbol{k}_F\|$ in $d>1$ render the Haldane (1992) bosonisation scheme in $d>1$ of limited validity.

A. The case of $d=1$

Here by choosing a positive direction, we consider one-dimensional *vectors* $\boldsymbol{q}$ and $\boldsymbol{k}$ as *scalars* and define $k_F{:=}\|\boldsymbol{k}_F\|$. We specifically consider the one-dimensional Luttinger model (Luttinger 1963, Mattis and Lieb 1965; see, e.g., Voit 1994) for spin-less fermions and in doing so replace S by $r=\pm$ to distinguish $g_S(\boldsymbol{q})\equiv g_r(\boldsymbol{q})$ associated with the branch of left ($r=-$) and right ($r=+$) movers. We have

$$g_r(\boldsymbol{q}){:=}\int_{-\lambda/2+rk_F+\boldsymbol{q}/2}^{\lambda/2+rk_F-\boldsymbol{q}/2}\mathrm{d}\boldsymbol{k}\,\big[\mathsf{n}_r(\boldsymbol{k}-\boldsymbol{q}/2)-\mathsf{n}_r(\boldsymbol{k}+\boldsymbol{q}/2)\big], \tag{40}$$

where λ is a cut-off parameter (see text preceding Eq. (3) above). One readily verifies that the choice of the boundaries of the integral on the RHS of Eq. (40) indeed restricts the domain of integration to $\bar{S}$.

Through decomposing the integral on the RHS of Eq. (40) into two integrals involving separately $\mathsf{n}_r(\boldsymbol{k}-\boldsymbol{q}/2)$ and $\mathsf{n}_r(\boldsymbol{k}+\boldsymbol{q}/2)$ and subsequently transforming the variable of integration $\boldsymbol{k}$ ($\boldsymbol{k}\rightharpoonup\boldsymbol{k}+\boldsymbol{q}/2$ in the first and $\boldsymbol{k}\rightharpoonup\boldsymbol{k}-\boldsymbol{q}/2$ in the second), cancellation of some contributions to the resulting integrals gives rise to $\int_{-\lambda/2+rk_F}^{-\lambda/2+rk_F+\boldsymbol{q}}\mathrm{d}\boldsymbol{k}\,\mathsf{n}_r(\boldsymbol{k})-\int_{\lambda/2+rk_F-\boldsymbol{q}}^{\lambda/2+rk_F}\mathrm{d}\boldsymbol{k}\,\mathsf{n}_r(\boldsymbol{k})$. For sufficiently large positive λ (see text following Eq. (3) above), $\mathsf{n}_r(\boldsymbol{k})$ in the first integral can be replaced by unity (zero) and in the second integral by zero (unity) for $r=+$ ($r=-$), thus in combination yielding

$$g_r(\boldsymbol{q})=r\,\boldsymbol{q}. \tag{41}$$

This result is in full conformity with that in Eq. (38). It is important that Eq. (41), which applies for 'sufficiently' large λ, is independent of λ itself. Since a correct treatment of the problem at hand requires that $\mathsf{n}_r(\boldsymbol{k})$ be cut off (in a 'soft' manner) for $|\boldsymbol{k}|$ larger than a certain cut-off value, say $\tilde{\lambda}_r>0$ (see Eq. (20) above and text following it), it is however necessary that $|-\lambda/2+rk_F|\ll\tilde{\lambda}_r$ ($|\lambda/2+rk_F|\ll\tilde{\lambda}_r$) for $r=+$ ($r=-$). Since $\tilde{\lambda}_r$ can be chosen arbitrarily large, we observe that in $d=1$ nothing stands in the way of choosing λ 'sufficiently' large and thus fulfilling Eq. (41).

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[21] This 'some error term' is in the work by Haldane (1992) — Eq. (34) herein — denoted by $X_\alpha(q,q')$. Haldane remarks (see p. 20 in Haldane 1992): "Because (in contrast to the original Tomonaga calculation in 1D) some of this ≪correction≫ involves states at the Fermi surface, this is perhaps not as innocuous an approximation in higher dimensions, but appears to be valid in the long-wavelength limit." In light of our considerations, it should be evident that "but appears to be valid in the long-wavelength limit" is not tenable. Note that full inclusion of $X_\alpha(q,q')$ destroys the Kac-Moody algebra.

[22] Houghton and Marston (1993) estimate (specifically for $d=3$) the size of 'some error term' to be on the order of $(\Lambda^{d-1}\lambda/\mathcal{N})(L/[2\pi])^d$, to be compared with that of the quantum anomaly which is on the order of $(\Lambda^d/\mathcal{N})(L/[2\pi])^d$. The choice $\lambda=\Lambda/\sqrt{\mathcal{N}}$ leads to the above-mentioned statement.

B. The case of $d = 2$

In the cylindrical polar coordinates $(\|\boldsymbol{k}\|, \theta)$, with $\boldsymbol{q}$ the polar axis (i.e. with $\boldsymbol{k} \cdot \boldsymbol{q} = \|\boldsymbol{k}\| \, \|\boldsymbol{q}\| \cos\theta$; for convenience we choose the origin of $\boldsymbol{k}$ to coincide with the centre of the Fermi sea) we have

$$g_S(\boldsymbol{q}) = \int_{\theta_{S;1}}^{\theta_{S;2}} \mathrm{d}\theta \, \big\{ j^-_{\boldsymbol{q}}(\theta) - j^+_{\boldsymbol{q}}(\theta) \big\}, \tag{42}$$

where

$$j^{\pm}_{\boldsymbol{q}}(\theta) := \int_{k_1(\theta)}^{k_2(\theta)} \mathrm{d}\|\boldsymbol{k}\| \, \|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta) \pm \boldsymbol{q}/2\big). \tag{43}$$

In Eq. (42), $\theta_{S;1}$ and $\theta_{S;2}$, $\theta_{S;1} < \theta_{S;2}$, specify the interval $(\theta_{S;1}, \theta_{S;2})$ associated with 'patch' S and in Eq. (43) $k_1(\theta)$ and $k_2(\theta)$, with $\theta \in (\theta_{S;1}, \theta_{S;2})$, specify the radial extent of 'squat box' $\bar{S}$ in the direction θ; we have $k_2(\theta) - k_1(\theta) = \lambda$ (see text preceding Eq. (3) above). Making use of $\|\boldsymbol{k} \pm \boldsymbol{q}/2\| \approx \|\boldsymbol{k}\| \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta$, which is appropriate for small $\|\boldsymbol{q}\|$, and some change of variables we obtain

$$j^{\pm}_{\boldsymbol{q}}(\theta) \approx \int_{k_1(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta}^{k_2(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta} \mathrm{d}\|\boldsymbol{k}\| \, \|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta)\big) \mp \frac{1}{2}\|\boldsymbol{q}\| \cos\theta \int_{k_1(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta}^{k_2(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta} \mathrm{d}\|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta)\big). \tag{44}$$

For generality we here take $\boldsymbol{k}_F$ to depend upon θ. With $k_1(\theta) < \|\boldsymbol{k}_F(\theta)\| < k_2(\theta)$ we assume $\mathsf{n}\big(\boldsymbol{k}(\theta)\big)$ to be regular in the neighbourhoods of $k_1(\theta)$ and $k_2(\theta)$. For sufficiently wide neighbourhoods which accommodate $k_j(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta$, $j = 1, 2$, for given values of $\|\boldsymbol{q}\|$ and θ, we have

$$\begin{aligned} \int_{k_1(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta}^{k_2(\theta) \pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta} \mathrm{d}\|\boldsymbol{k}\| \, \|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta)\big) &= \int_{k_1(\theta)}^{k_2(\theta)} \mathrm{d}\|\boldsymbol{k}\| \, \|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta)\big) \\ &\pm \frac{1}{2}\|\boldsymbol{q}\| \cos\theta \Big\{ k_2(\theta) \mathsf{n}\big(\boldsymbol{k}_2(\theta)\big) - k_1(\theta) \mathsf{n}\big(\boldsymbol{k}_1(\theta)\big) \Big\} + \mathcal{O}(\|\boldsymbol{q}\|^2), \end{aligned} \tag{45}$$

from which we obtain

$$j^-_{\boldsymbol{q}}(\theta) - j^+_{\boldsymbol{q}}(\theta) = \|\boldsymbol{q}\| \cos\theta \Big\{ k_1(\theta) \mathsf{n}\big(\boldsymbol{k}_1(\theta)\big) - k_2(\theta) \mathsf{n}\big(\boldsymbol{k}_2(\theta)\big) + \int_{k_1(\theta)}^{k_2(\theta)} \mathrm{d}\|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\theta)\big) \Big\} + \mathcal{O}(\|\boldsymbol{q}\|^2). \tag{46}$$

The last term on the RHS enclosed by braces makes explicit that in $d = 2$ (in fact in any $d > 1$) it is not only $\mathsf{n}(\boldsymbol{k})$ far inside and far outside the FS that is relevant but that the behaviour of $\mathsf{n}(\boldsymbol{k})$ over the entire 'squat box' is significant.

Owing to the continuity of the RHS of Eq. (46), as a function of θ, the first mean-value theorem (Whittaker and Watson 1927, p. 65) applies and we can write

$$g_S(\boldsymbol{q}) = \big(\theta_{S;2} - \theta_{S;1}\big) \Big\{ j^-_{\boldsymbol{q}}(\tilde{\theta}_S) - j^+_{\boldsymbol{q}}(\tilde{\theta}_S) \Big\}, \quad \text{for } \textit{some } \tilde{\theta}_S \in [\theta_{S;1}, \theta_{S;2}]. \tag{47}$$

For the sake of argument let us now make an appeal to 'common wisdom' and make use of the trapezoidal rule (Abramowitz and Stegun 1972, p. 885)

$$\int_a^{a+\Delta} \mathrm{d}x \, f(x) = \frac{\Delta}{2}\big[f(a + \Delta) + f(a) \big] + \mathcal{O}(|\Delta|^3); \tag{48}$$

neglecting the rest term we have

$$\int_{k_1(\tilde{\theta}_S)}^{k_2(\tilde{\theta}_S)} \mathrm{d}\|\boldsymbol{k}\| \, \mathsf{n}\big(\boldsymbol{k}(\tilde{\theta}_S)\big) \approx \frac{1}{2}\big(k_2(\tilde{\theta}_S) - k_1(\tilde{\theta}_S)\big) \big[\mathsf{n}\big(\boldsymbol{k}_2(\tilde{\theta}_S)\big) + \mathsf{n}\big(\boldsymbol{k}_1(\tilde{\theta}_S)\big) \big]. \tag{49}$$

Making use of the properties [23] $\mathsf{n}\big(\boldsymbol{k}_1(\tilde{\theta}_S)\big) = 1$ and $\mathsf{n}\big(\boldsymbol{k}_2(\tilde{\theta}_S)\big) = 0$ (see texts following Eqs. (37) and (40) above), from Eq. (49) we obtain

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[23] For interacting particles, $\mathsf{n}(k)$ can significantly differ from unity (zero) even for k far inside (outside) the Fermi sea (for some recent quantum-Monte-Carlo based results concerning $\mathsf{n}(k)$ see Moroni, Senatore and Fantoni 1997). Therefore, it would be advisable to leave $\mathsf{n}(k_1)$ and $\mathsf{n}(k_2)$ in the formalism and determine them in a self-consistent manner. In this connection note that in applying the Haldane (1992) bosonisation scheme, the inequalities $\lambda \ll \Lambda \ll \|k_F\|$ are to be satisfied and a small λ does not allow k to be either far inside or far outside the Fermi sea.

$$g_S(\boldsymbol{q}) \approx \frac{1}{2}\|\boldsymbol{q}\|\cos(\tilde{\theta}_S)\big(\theta_{S;2}-\theta_{S;1}\big)\,\big[k_1(\tilde{\theta}_S)+k_2(\tilde{\theta}_S)\big] = \Lambda\,\boldsymbol{q}\cdot\boldsymbol{n}_S, \tag{50}$$

where we have made use of the fact that $\big[k_1(\tilde{\theta}_S)+k_2(\tilde{\theta}_S)\big]/2=:\bar{k}(\tilde{\theta}_S)$ describes the distance from the centre of the Fermi sea to 'patch' S of the FS at angel $\tilde{\theta}_S$ (for $\tilde{\theta}_S=(\theta_{S;1}+\theta_{S;2})/2$, $\bar{k}(\tilde{\theta}_S)=\|\boldsymbol{k}_F\|$) and that for small $(\theta_{S;2}-\theta_{S;1})$, $\bar{k}(\tilde{\theta}_S)\times(\theta_{S;2}-\theta_{S;1})=\Lambda$, the length of the 'patch' (see text preceding Eq. (3) above). Further, we have employed $\|\boldsymbol{q}\|\cos(\tilde{\theta}_S)=\boldsymbol{q}\cdot\boldsymbol{n}_S$. The result as presented on the RHS of Eq. (50) is in full conformity with the RHS of Eq. (38).

Through the above exercise we have determined the set of assumptions which are needed in order to obtain the Kac-Moody algebra in Eq. (39) in $d=2$. We now consider in how far our reliance upon 'common wisdom' is justified. An aspect that immediately is noticed is the fact that $\mathcal{O}(|\Delta|^3)$ on the RHS of Eq. (48) is proportional to $\mathrm{d}^2f(x)/\mathrm{d}x^2$ for *some* x in the open interval $(a,a+\Delta)$, implying that for a function $f(x)$ which is not twice continuously differentiable, [24] there exists *no* reason, whatever, that the RHS of Eq. (48) would be in any way related to the left-hand side. It is well-known that for FLs $\mathsf{n}(\boldsymbol{k})$ is discontinuous at $\boldsymbol{k}=\boldsymbol{k}_F$ and for LLs, $\mathsf{n}(\boldsymbol{k})$ though continuous at $\boldsymbol{k}=\boldsymbol{k}_F$, is *not* differentiable at this point. It is interesting, however, that independent of how one proceeds from Eq. (46) to the final result, the term $\|\boldsymbol{q}\|\cos\theta\equiv\boldsymbol{q}\cdot\boldsymbol{n}_S$ remains unchanged so that the correct treatment of Eq. (47) does *not* lead to destruction of the Kac-Moody algebra. Nevertheless, here, contrary to the $d=1$ case, λ *cannot* be chosen arbitrarily large, but must satisfy the requirement $\lambda\ll\Lambda$. As we shall demonstrate below, this restriction gives rise to a significant effect.

Owing to $\Lambda\ll\|\boldsymbol{k}_F\|$, the restriction $\lambda\ll\Lambda$ justifies use of the following two approximations: $k_1(\theta)\equiv\|\boldsymbol{k}_F\|-\lambda/2\approx\|\boldsymbol{k}_F\|$ and $k_2(\theta)\equiv\|\boldsymbol{k}_F\|+\lambda/2\approx\|\boldsymbol{k}_F\|$. By the same reasoning we can employ $\mathsf{n}\big(\boldsymbol{k}_1(\theta)\big)-\mathsf{n}\big(\boldsymbol{k}_2(\theta)\big)\approx Z_{\boldsymbol{k}_F}$. Further, since $k_2(\theta)-k_1(\theta)=\lambda$ and $0\leq\mathsf{n}(\boldsymbol{k})\leq 1$ (but $\mathsf{n}(\boldsymbol{k})\not\equiv 0$) for *all* $\boldsymbol{k}$, we have

$$0<\int_{k_1(\theta)}^{k_2(\theta)}\mathrm{d}\|\boldsymbol{k}\|\;\mathsf{n}\big(\boldsymbol{k}(\theta)\big)<\lambda. \tag{51}$$

Thus as long as $Z_{\boldsymbol{k}_F}\gg\lambda/\|\boldsymbol{k}_F\|$, we can neglect the integral on the RHS of Eq. (46) and using the above approximation for $\mathsf{n}\big(\boldsymbol{k}_1(\theta)\big)-\mathsf{n}\big(\boldsymbol{k}_2(\theta)\big)$ write

$$g_S(\boldsymbol{q})\approx\zeta^{-1}\,\Lambda\,\boldsymbol{q}\cdot\boldsymbol{n}_S, \tag{52}$$

where $\zeta^{-1}:=Z_{\boldsymbol{k}_F}$, to be compared with the expression in Eq. (50). Consequently, the Kac-Moody algebra associated the $g_S(\boldsymbol{q})$ in Eq. (52) is similar to that in Eq. (39), however with the RHS multiplied by $\zeta^{-1}\equiv Z_{\boldsymbol{k}_F}$. Since $\boldsymbol{q}$ is reciprocal to $\boldsymbol{x}$, scaling of the RHS of Eq. (39) by ζ^{-1} (here $Z_{\boldsymbol{k}_F}$), gives rise to scaling of $\boldsymbol{x}$, and by the requirement of the Lorentz invariance, of t, by ζ. Thus with the RHS of Eq. (39) being multiplied by ζ^{-1}, we have (see Eq. (5) above)

$$G_{0;\mathrm{f}}(\boldsymbol{k},t;S)\rightharpoonup G_{0;\mathrm{f}}(\boldsymbol{k},t;S)+(1-\zeta)\mathcal{F}_{\boldsymbol{k},\omega}\Big[\frac{\Lambda}{(2\pi)^2}\,\frac{\exp\big(i\boldsymbol{k}_F(S)\cdot\boldsymbol{x}\big)}{\zeta\,\boldsymbol{x}\cdot\boldsymbol{n}_S-\zeta\,t-i\,\eta\,\mathrm{sgn}(\zeta t)}\Big],\;\eta\downarrow 0. \tag{53}$$

It can be verified that this modification is 'harmless' as far as the fundamental question with regard to *nature* of the metallic state of the system under consideration is concerned (i.e. whether it is a FL or otherwise).

The situation undergoes a fundamental change when $Z_{\boldsymbol{k}_F}$ is small, on the order of $\lambda/\|\boldsymbol{k}_F\|$ or smaller, or vanishing, as is the case for such NFLs as marginal FLs and LLs. In such cases, the magnitude of the RHS of Eq. (46), i.e. $\zeta^{-1}\|\boldsymbol{k}_F\|$, is on the order of λ (see Eq. (51) above) from which it follows that the associated Kac-Moody algebra is inadequate: since $\zeta^{-1}\approx\lambda/\|\boldsymbol{k}_F\|\equiv 1/\mathcal{N}$ (see paragraph following Eq. (39) above), in the case at hand the size of the quantum anomaly is by on the order of $1/\sqrt{\mathcal{N}}$ *smaller* than that of 'some error term' (see footnote 22).

It has now become evident why, as we have established above through explicit calculation (see Eq. (22) and the following text), within the Haldane (1992) bosonisation scheme, the large-$\|\boldsymbol{x}\|$ behaviour of $-iG(\boldsymbol{x},t=0^-)$ is always FL-like: the Kac-Moody algebra in Eq. (39) is only compatible with metallic states whose corresponding $Z_{\boldsymbol{k}_F}\gg\lambda/\|\boldsymbol{k}_F\|$. This compatibility does not guarantee that Eq. (39) gives a sufficiently complete account of the physical processes at low energies of metallic systems for which holds $Z_{\boldsymbol{k}_F}\gg\lambda/\|\boldsymbol{k}_F\|$, since our explicit calculations show that under *all* circumstances $\Sigma(0)=0$, which is highly uncommon.

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[24] To be explicit, the rest term on the RHS of Eq. (48) is equal to $(-\Delta^3/12)f''(x)$, with $x\in(a,a+\Delta)$.

We note in passing that the result in Eq. (38) is exact, for *all* d, for the case where the coupling constant of the particle-particle interaction is vanishing. From this perspective, our finding that within the framework of the Haldane (1992) bosonisation scheme $\Sigma(0) = 0$ under *all* conditions, is consistent with the fact that for non-interacting particles $\Sigma(\boldsymbol{k}, \omega)$ is identically vanishing.

In view of the above-indicated shortcoming of the Haldane bosonisation program in $d > 1$, we are presently not in a position to make *any* well-founded judgement with regard to the true nature of the metallic state of the model under discussion. By taking into account the fact that an approximation for $\bar{\chi}_0(x)$, of the type presented in Eq. (26), innocuous as it may seem at first glance, is capable of turning a "FL" into a "LL", we are of the opinion that the model considered here, may as yet have some surprising properties in stores, to be exposed by means of a future refined treatment.

VI. A BRIEF DISCUSSION OF SOME ALTERNATIVE BOSONISATION PROGRAMMES

In light of our finding with regard to a major shortcoming in the Haldane (1992) bosonisation scheme in $d > 1$, it is important to mention that in our considerations we have been dealing with a specific realisation of Haldane's scheme, namely one that has been explicitly dealt with by Haldane (1992) and Houghton and Marston (1993). In addition to this, two alternative realisations of this scheme have been put forward, one by Castro Neto and Fradkin (1994a,b, 1995) and the other by Kopietz and Schönhammer (1996).

In the realisation by Castro Neto and Fradkin (1994a,b, 1995), the authors construct a coherent-state path-integral formalism for the bosonised problem which concerns low-energy long-wavelength excitations of the fermionic problem. The coherent states in this approach are eigenstates of the annihilation operators $\{\hat{a}_{\boldsymbol{q}}(\boldsymbol{k})\}$ which are constructed from the equal-time density operators and which under some assumptions, similar in spirit to those in the works by Haldane (1992) and Houghton and Marston (1993), the authors demonstrate to form a generalised Kac-Moody algebra. An aspect which is crucial to arriving at this result concerns replacing the density *operator* in the momentum space with the *expectation value* of it with respect to the ground state of the *non-interacting* fermion system. We can demonstrate (Farid 1999b) that this approximation is at odds with a LL as well as some unconventional FL metallic states in $d > 1$, even though, as Castro Neto and Fradkin have shown (see Appendix B in Castro Neto and Fradkin 1994b), it is compatible with the former in $d = 1$; remarkably, the distinction between the two cases, corresponding to $d = 1$ and $d > 1$, turns out to arise from the same mechanism that renders Eq. (39) (or Eq. (38)) compatible with a LL metallic state in $d = 1$ but incompatible in $d > 1$.

In the approach by Kopietz and Schönhammer (1996) (for a detailed exposition see Kopietz 1997) the fermion problem is described in terms of an imaginary-time functional integral over Grassmann fields. The action in this description is subsequently related to a Hamiltonian which is subjected to Haldane's (1992) 'patching' scheme of the FS. Following this, the two-body interaction is transformed away by means of a Hubbard-Stratonovich transformation. Through application of a second Hubbard-Stratonovich transformation, a composite Grassmann field is eliminated in the formalism in exchange for a collective bosonic field. The kinetic term of the effective action thus obtained, is subsequently dealt with through application of the perturbation theory. Within the framework of the Gaussian approximation, all terms beyond the second order in this kinetic contribution are neglected. Kopietz, Hermisson and Schönhammer (1996) find justification for the Gaussian approximation in the so-called generalised closed-loop theorem (or generalised loop-cancellation theorem), applicable to all spatial dimensions [25] d. The proof of this theorem in $d > 1$ is based upon two simplifying approximations, i) the so-called 'diagonal-patch' approximation [26] and ii) local linearisation of the energy dispersion of the non-interacting fermions. For this theorem to be significant for applications, the effective screened interaction, in momentum space, amongst particles must be negligible for momenta $\boldsymbol{q}$ whose magnitudes $\|\boldsymbol{q}\|$ are larger than a cut-off q_c, with q_c assumed to satisfy $q_c \ll \|\boldsymbol{k}_F\|$. On the basis of the generalised closed-loop theorem, it has been shown that the just-mentioned Gaussian approximation becomes asymptotically exact for $q_c/\|\boldsymbol{k}_F\| \to 0$. Validity of neither of the two mentioned conditions is dependent upon an equivalent of Eq. (39). Since in the limit of long wavelengths and high density of fermions the theory based upon the Gaussian approximation has been shown to reduce to those by Houghton and Marston (1993) and Castro Neto and Fradkin (1994a,b) (see specifically § 4.2.5 in Kopietz 1997), we may conclude that validity of the generalised

[25] For $d = 1$ the loop-cancellation theorem is due to Dzyaloshinskiĭ and Larkin (1974). For an extensive elaboration on this theorem, in $d = 1$, see (Bohr 1981).

[26] This is equivalent to the approximation as expressed through Eqs. (34) and (35).

closed-loop theorem in $d > 1$ must be restricted to FLs. That this may be the case can be made plausible (Farid 1999b) by considering the fact that validity of the generalised closed-loop theorem depends on the validity of the so-called 'asymptotic velocity conservation' (see text following Eq. (55) below) in $d > 1$ (this 'velocity conservation' is *exact* in $d = 1$ for much the same reason that Eq. (39) is of general applicability in $d = 1$ but *not* in $d > 1$); for NFLs, since the Fermi velocity is by definition undefined, [27] one encounters a fundamental difficulty in rigorously demonstrating the asymptotic validity of the generalised closed-loop theorem for cases where the metallic state is a NFL. The exactness of the loop-cancellation theorem in $d = 1$, irrespective of the nature of the metallic state of the system, is directly related to existence of *exact* (as opposed to asymptotically-correct) conservation laws, which embody the well-known Ward identities (see, e.g., Bohr 1981). To illustrate our line of reasoning, we mention that the essence of the 'asymptotic velocity conversation' is expressed in the following result (see, e.g., Metzner, Castellani and Di Castro 1998):

$$G_0(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)\,G_0(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2) = \frac{G_0(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)-G_0(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2)}{\omega_0-\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}}+\mathcal{O}(\|\boldsymbol{q}\|^m)}, \tag{54}$$

where $\boldsymbol{v}^0_{\boldsymbol{p}}:=\boldsymbol{\nabla}_{\boldsymbol{p}}\omega^0_{\boldsymbol{p}}$ and m stands for an *integer larger than unity*; for *linear* dispersions, $\mathcal{O}(\|\boldsymbol{q}\|^m)$ is vanishing. It can be trivially shown that

$$G(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)\,G(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2) = \frac{G(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)-G(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2)}{\omega_0-\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}}-[\Sigma(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2)-\Sigma(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)]+\mathcal{O}(\|\boldsymbol{q}\|^m)}. \tag{55}$$

The 'asymptotic velocity conservation' to which we have referred above, reflects the fact that denominator of the expression on the RHS of Eq. (54) asymptotically behaves like $(\omega_0-\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}})$ for $\|\boldsymbol{q}\|\to 0$. This can be shown to be the case also for the denominator of the expression on the RHS of Eq. (55) *provided* the system under consideration is a FL; for NFLs, $[\Sigma(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2)-\Sigma(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)]$ is the dominant contribution as compared with $(\omega_0-\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}})$ and therefore for NFLs the RHS of Eq. (55) *cannot* be brought into a form resembling the RHS of Eq. (54). This amounts to the fact that the route to demonstrating the asymptotic validity of the closed-loop theorem on the basis of a perturbation series in terms of the *interacting* Green function is closed when the system under consideration is not a FL, supporting the point of view that implications of this theorem may not be self-consistent. It is important to note that $\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}}$ in denominators of the expressions on the RHSs of Eqs. (54) and (55) are of the same origin as $\boldsymbol{q}\cdot\boldsymbol{n}_S$ on the RHS of Eq. (39) — recall that since FS 'patches' are flat, $\boldsymbol{v}^0_{\boldsymbol{p}}$, with $\boldsymbol{p}\in S$, is parallel to $\boldsymbol{n}_S$. Note also that dominance of $[\Sigma(\boldsymbol{p}+\boldsymbol{q}/2,\omega+\omega_0/2)-\Sigma(\boldsymbol{p}-\boldsymbol{q}/2,\omega-\omega_0/2)]$ in comparison with $(\omega_0-\boldsymbol{q}\cdot\boldsymbol{v}^0_{\boldsymbol{p}})$, in cases of NFLs, is related to that of the suppressed terms on the RHS of Eq. (39) when NFLs are concerned.

For completeness, we mention that Kopietz and Castilla (1996) have introduced a method of incorporating the non-linear corrections to the non-interacting energy dispersions and Kopietz (1997) has generalised the original formalism through which corrections beyond the Gaussian approximation can be accounted for.

VII. SUMMARY AND CONCLUDING REMARKS

In this work we have first employed a method of characterising metallic states of systems of interacting fermions as FLs (including conventional and un-conventional ones) and NFLs in terms of the asymptotic behaviour of $\delta G(\omega)$ for small ω. Following a close examination of some results in the energy-momentum domain obtained by earlier authors concerning a model of interacting fermions in two spatial dimensions, which proved inconclusive as regards the true metallic state of this model, we have studied (full details to be published elsewhere) some specific aspects of the single-particle Green function in the space-time domain. This study has confronted us with an inconsistency (namely that under *all* conditions metallic states of systems in $d > 1$ must be unconventional FLs) which we have subsequently traced back to a fundamental shortcoming in an operator algebra that stands central to the process of bosonisation of the low-energy degrees of freedom of metallic systems of fermions in $d > 1$ spatial dimensions. We have explicitly

[27] This follows the fact that in metallic NFLs, $\Sigma(k,\omega_F)$ and/or $\Sigma(k_F,\omega)$ fail, by definition, to be continuously differentiable in neighbourhoods of $k = k_F$ and $\omega = \omega_F$, respectively (see text preceding Eq. (1) above).

shown that this shortcoming is due to inapplicability in $d > 1$ of an underlying assumption when NFLs, specifically LLs, are concerned. [28] On the other hand, since *all* metallic states according to the Haldane bosonisation scheme turn out to be unconventional FLs, it follows that even in the case of FLs the mentioned operator algebra does not describe the underlying physical processes sufficiently accurately. It *may* turn out that the mentioned shortcoming can be removed without needing to destroy the fundamental structure of the operator algebra, i.e. the Kac-Moody algebra, to which we have just referred. From these observations we have arrived at the conclusion that on the basis of the available results *nothing* can be inferred as regards the true metallic state of the two-dimensional model system under consideration. □

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